# 2018 Manitoba Mathematical Competition Solutions

1. (a) Find an integer greater than 1 that leaves remainder 1 when divided by each of 2, 3, 4, 5 and 6.

   (b) If $a^2 - b^2 = 42$ and $2a + 2b = 14$, find $3a - 3b$.

**Solution:**

(a) The least common multiple of 2, 3, 4, 5 and 6 is 60. So one answer is 61.

(b) $(a+b)(a-b) = 42$ and $a+b = 7$, so $a-b = 6$ and $3a-3b = 3(a-b) = 3 \cdot 6 = 18$.

**Comments:**

- To solve(a), add one to <u>any</u> multiple of 60.

2. (a) Solve for $x$:

$$x\sqrt{x+5}+5\sqrt{x+5} \;=\; 8$$

(b) Solve for $a$ and $b$:

$$\begin{aligned} a+b &= 53 \\ \sqrt{a}-\sqrt{b} &= 5 \end{aligned}$$

**Solution:**

(a) $(x+5)\sqrt{x+5}=(x+5)^{\frac{3}{2}}=8$. So $x+5=8^{\frac{2}{3}}=4$. Therefore $x=-1$.

(b) $\sqrt{b}=\sqrt{a}-5$, so

$$\begin{aligned} a+(\sqrt{a}-5)^2 &= 53 \\ a+a-10\sqrt{a}+25 &= 53 \\ 2a-10\sqrt{a}-28 &= 0 \\ a-5\sqrt{a}-14 &= 0 \\ (\sqrt{a}-7)(\sqrt{a}+2) &= 0 \end{aligned}$$

We have $\sqrt{a}>0$ so $\sqrt{a}=7$ and $a=49$. Now $b=53-49=4$ (and directly verify that the second equation is satisfied). So $(a,b)=(49,4)$.

**Comments:**

- Alternative approach to (b): square the second equation to get $a+b-2\sqrt{ab}=25$. Subtracting from the first gives $2\sqrt{ab}=28$, so $ab=14^2=196$. So $a$ and $b$ are the roots of $t^2-53t+196=0$, which are (by factorization or the QF—the latter being a non-trivial by-hand calculation!) 49 and 4. By the second equation $a>b$ so $(a,b)=(49,4)$.
- A number of inappropriate approaches to (b) happened to "work" insofar as getting the answer. Most commonly many assumed incorrectly that the second equation forces $a$ and $b$ to be perfect squares. Many used guess-and-check, stopping when a solution is found. First, this is an inefficient approach (especially considering that one cannot assume even that there are finitely many cases to check!); second, there is nothing in the statement of the problem that precludes multiple solutions, in which case this strategy would not solve the problem. In general, a deductive approach is preferred; guess-and-check is more appropriate for <u>finding</u> solutions than for <u>presenting</u> them. If progress can be made by approaching a problem deductively this is <u>always</u> preferred over guess-and-check or dividing into cases; when the latter strategies turn out to be necessary, analytic pre-processing generally reduces the work and increases the likelihood of success.
- Many wrong answers to (b) began with the fallacy some call the "universal homomorphism"—in this case inferring from $\sqrt{a}-\sqrt{b}=5$ that we must have $a-b=25$. The universal homomorphism falsely asserts that $f(x*y)=f(x)*f(y)$ where $f$ is any function and $*$ is any operation. It is a serious, and common, error that should be curtailed before it becomes a habit.

3. (a) What is the sum of the digits of $10^{50} - 5^5$?

(b) Evaluate the sum $f(1) + f(2) + f(3) + \cdots + f(100)$, where $f(k) = \dfrac{1}{4k^2 - 1}$.

**Solution:**

(a) The sum of the digits of $10^{50} - 3125$ is, directly, $46(9) + 6 + 8 + 7 + 5 = 440$.

(b) First write $\frac{1}{4x^2-1}$ as $\frac{1}{2}\left[\frac{1}{2x-1} - \frac{1}{2x+1}\right]$. Then the sum telescopes as:

$$\frac{1}{2}\left[\left(1 - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{199} - \frac{1}{201}\right)\right] = \frac{1}{2}\left[1 - \frac{1}{201}\right] = \frac{100}{201}.$$

## Comments:

- Concerning (b): we don't expect all students to instantly recognize a telescoping series, though it is always advisable for students, when exploring, to consider whether a prominent difference of squares contains some key. However, students who bothered to replace 100 with 1, 2, 3 would have been rewarded with sums $\frac{1}{3}, \frac{2}{5}, \frac{3}{7}$ which should suggest the formula $f(1) + \cdots + f(n) = \frac{n}{2n+1}$. Adding $f(n+1)$ would confirm this by yielding $\frac{n+1}{2n+3}$. It is very common for contest problems with large numbers to yield to exploration involving corresponding questions using smaller numbers. Too many students tried to jump to a complete answer before sufficient exploration to understand the behaviour of this sum.

4. (a) Determine the sum of all five-digit numbers whose digits consist only of 2s or 3s.

   (b) Define a sequence of real numbers by $x_0 = 1$ and $x_{k+1} = \dfrac{2x_k - 4}{x_k + 2}$ for all $k \geq 0$. Determine the value of $x_{2018}$.

**Solution:**

(a) There will be $2^5 = 32$ such numbers. Consider the $10^k$s digits. Half will be 2s and the other 16 will be 3s. So to add up the $10^k$ digits column we will have $16(2+3) = 80$. This occurs for each of the five digit columns. So the sum is equal to $80 \cdot 11111 = 888880$.

(b) Simple experimentation gives $x_{k+2} = \frac{-4}{x_k}$ and so $x_{k+4} = x_k$. Thus $x_{2016} = x_0 = 1$ and so $x_{2018} = \frac{-4}{x_{2016}} = -4$.

**Comments:**

- In (a) many students found the sum of all the digits. The problem did not ask for this, and it seems likely they were pattern-matching the problem without reading carefully, and assuming that it matched something else they'd done recently, perhaps in preparation for competition. While there are commonalities among contest-style problems it is also true that contest questions often contain slight variants of classical problems. For this reason very careful reading of questions before attempting them is essential.
- (b) is another illustration of the value in trying out small cases when faced with daunting questions involving large numbers. Our analytic writeup here may obscure this point. A student who does this will find that $x_2 = -4$ and $x_4 = x_0$. It follows (not by inductive reasoning, but deductively!) that $x_{k+4} = x_k$ for all $k$ and the result follows easily as in our solution.

5. (a) For how many positive integers $n$ is $\frac{n}{3}$ a three-digit integer and $6n$ a four-digit integer?

(b) Suppose $a, b, c$ are nonzero numbers and $ax + a = by + b = cz + c = a + b + c$. Use this information to find the numerical value of the expression $xyz-(x+y+z)$.

**Solution:**

(a) $100 \le \frac{n}{3} \le 999$ so $n$ is a multiple of 3 in the range $300 \le n \le 2997$. Further, $1000 \le 6n \le 9999$ so $166 < n < 1667$. The overlap between these sets is $300 \le n \le 1666$. So the required number is equal to the number of multiples of 3 in this range which is 456.

(b) $x = \frac{b+c}{a}$, $y = \frac{a+c}{b}$ and $z = \frac{a+b}{c}$. So

$$\begin{aligned} xyz-(x+y+z) &= \frac{(b+c)(a+c)(a+b) - b^2c - bc^2 - a^2c - ac^2 - a^2b - ab^2}{abc} \\ &= \frac{a^2b + ab^2 + abc + b^2c + a^2c + abc + c^2a + c^2b - b^2c - \cdots - ab^2}{abc} \\ &= \frac{2abc}{abc} = 2 \text{ (since } a, b, c \neq 0) \end{aligned}$$

**Comments:**

- A significant number of students appeared to regard part (a) as two distinct questions. Weak mathematical reading comprehension seems to be impairing an increasing number of students. Once more this underscores the importance of ensuring that you understand a question before attempting to solve it, and of not making unwarranted assumptions.
- The most common mathematical error in (a) was to miscount by 1 or 2, probably because of being unsure how to handle the boundary conditions or struggling with the distinction between inclusive and exclusive counting.
- A few papers found the answer for (b) by simply substituting values for $a, b, c$. For example $a = b = c = 1$ gives $x = y = z = 2$ so $xyz - (x + y + z) = 8 - 6 = 2$. This is a clever idea which might be considered correct given the sloppy choice of words in the question implying that there is a unique numerical value that is independent of the particular values of the variables. So any choices will lead to the correct answer. However, this is not a "given". It would be consistent with this wording (but also sloppy) if, in fact, there were distinctly different answers depending on, say, the sign of $abc$, and the answer required a statement of such cases. It may even have been that the answer could be as some formula involving $a, b, c$ In any case, this numerical shortcut does not comprise a logically complete solution. For full-solution contest questions markers are looking for solutions that fully demonstrate one's answer to be correct—even when a question doesn't explicitly ask for a proof—simply producing the correct answer is not generally worth full credit.

6. (a) Prove that, in any set $\{a_1, a_2, a_3, a_4\}$ of four integers, there are two elements that differ by a multiple of 3.

   (b) Prove that no positive integer has a square of the form $100k + 51$.

**Solution:**

(a) Every integer is of the form $3k, 3k + 1$ or $3k + 2$. By the Pigeonhole Principle two out of any four integers must have one of these forms. These will differ by a multiple of 3.

(b) Even squares are of the form $4k^2$ and odd squares are of the form $(2k + 1)^2 = 4(k^2 + k) + 1$. Thus perfect squares are either divisible by 4 or leave remainder 1 when divided by 4. $100k + 51 = 4(25k + 12) + 3$ leaves a remainder of 3 when divided by 4, so it cannot be a perfect square.

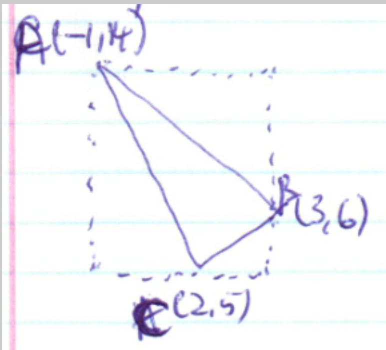
**Comments:**

- Stating the solution for (a) in terms of modular arithmetic makes it less awkward, but we do not assume students are familiar with this approach.
- Alternatively (b) can be solved a number of ways by explicitly handing digits of a square, which is facilitated by the factor 100 in the first term of the form. Students familiar with modular arithmetic can find a number of alternative pathways to the solution, none of which are notably easier than the approach shown, but might be articulated somewhat more elegantly.

7. The line $2x + y - 12 = 0$ intersects the parabola $y = x^2 - 4x + 9$ at points $A$ and $B$. If $C$ is the vertex of the parabola, find the area of $\triangle ABC$.

**Solution:** $y = (x-2)^2 + 5$ has vertex $C(2,5)$. Solving $y = x^2 - 4x + 9$ and $y = -2x + 12$ gives $x = 3$ or $x = -1$, with respectively, $y = 6$ or $14$, so the intersection points are $A(-1,14)$ and $B(3,6)$.

Method 1: the dotted rectangle has area $(14-5)(3-(-1)) = 36$ and the three right triangles have total area $\frac{1}{2}(2+1)(14-5) + \frac{1}{2}(3+1)(14-6) + \frac{1}{2}(3-2)(6-5) = 30$ so $\triangle ABC$ has area $36 - 30 = 6$



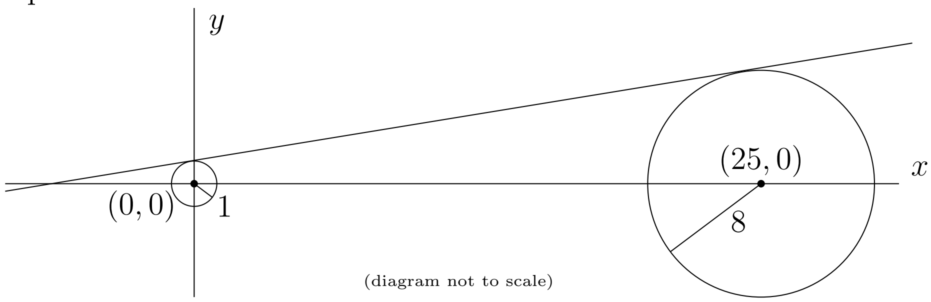

Method 2: The perpendicular distance from $C$ to line $AB$ is (standard formula) $\frac{|2(2)+5-12|}{\sqrt{4+1}} = \frac{3}{\sqrt{5}}$ and $AB = \sqrt{(3+1)^2 + (6+4)^2} = \sqrt{80} = 4\sqrt{5}$ and the required area is $\frac{1}{2} \cdot 4\sqrt{5} \cdot \frac{3}{\sqrt{5}} = 6$.

Method 3: $\triangle ABC = \frac{1}{2}\begin{vmatrix} 2 & 5 & 1 \\ 3 & 6 & 1 \\ -1 & 14 & 1 \end{vmatrix} = 6$ (classical formula not often taught anymore)

**Comments:**

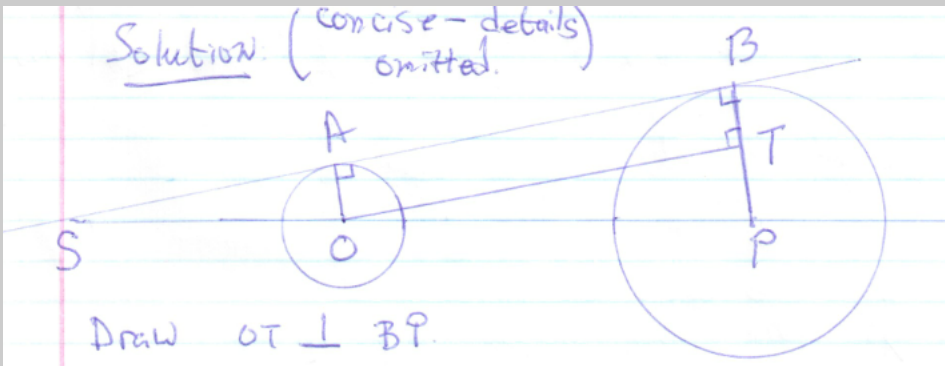
- Method 3 is for those familiar with determinants. Yet another way to use determinants: a $2 \times 2$ determinant gives the area of a parallelogram determined by its row vectors. So a fast solution that doesn't require knowledge of the old $3 \times 3$ formula but does require first-year linear algebra (which some HS students will know) would be to use $\vec{AB}$ and $\vec{AC}$ to give area $\frac{1}{2}\text{abs}\left(\begin{vmatrix} 4 & -8 \\ 3 & -9 \end{vmatrix}\right) = \text{abs}\left(\frac{-36+24}{2}\right) = 6$.
- Heron's formula and the distance formula between points on the plane gives another possible, though ugly, solution. Yes, it can be simplified all the way by hand.
- A few papers used the following simple device to find the area of the triangle: cut with a vertical line through $C$ (which meets $AB$ at $D(2,8)$). With $CD$ as a base find (and add) the areas of the two resulting triangles, whose altitudes at $A$ and $B$ are immediate.
- The original hand-drawn diagram here will eventually be replaced with a proper vector-graphic diagram.

8. A circle with radius 1 has centre at $(0, 0)$. A second circle with radius 8 has centre at $(25, 0)$. A line lies above both circles and is tangent to both, as illustrated. Find an equation for this line.




(diagram not to scale)

**Solution:**




(Sketch) Draw $OT \perp BP$. $OP = 25, TP = 8 - 1 = 7$, so by Pythagoras' Theorem, $OT = 24$. $\triangle SAO$ is similar to $\triangle SBP$ and to $\triangle OTP$. Therefore,

$$\frac{SO}{AO} = \frac{OP}{TP} = \frac{25}{7}.$$

Thus,

$$SO = \frac{25}{7} AO = \frac{25}{7}.$$

$S$ is the point $\left(\frac{-25}{7}, 0\right)$. The slope of $AB$ is $\tan \angle BSP = \tan \angle TOP = \frac{7}{24}$. So the equation of $AB$ is $y = \frac{7}{24}\left(x + \frac{25}{7}\right)$ or $7x - 24y + 25 = 0$.

**Alternate #1:** Another approach without the Ahaaa!: Enlarging by factor of 8 around $S$ maps the small circle to the large one. Thus $SO/(SO + 25) = 1/8$ which gives $SO = 25/7$ and $S$ is the point $\left(\frac{-25}{7}, 0\right)$. Thus $SA = \sqrt{SO^2 - 1} = 24/7$ and the slope of the line is $\tan(BSP) = \tan(ASO) = 1/SA = 7/24$ and the result follows by the point-slope formula.

## Comments:

- Another approach sets $SO = x$ then from similar triangles $(x+1)/1 = (x+26)/8$ yielding $x = 18/7$, with $x$-intercept of the line, $x = -25/7$.
- Again, when we have a chance the hand-drawn diagram in the solution will be replaced with a proper graphic.

9. If $p$ and $q$ are <u>consecutive</u> prime numbers, both greater than 3, prove that $p+q$ has at least 6 distinct positive divisors. (For example, $31+37=68$ which is divisible by $1, 2, 4, 17, 34$ and $68$.)

**Solution:** Since $p$ and $q$ are both odd, their average, $\frac{p+q}{2}=t$ is an integer. Since $t$ is strictly between two consecutive primes, it is composite.
So $t$ either has 2 or more distinct prime powers or $t$ is a power of a prime.

Case 1: Suppose $t$ has at least two distinct prime divisors $r_1$ and $r_2$. If these are both odd, then the divisors of $2t=p+q$ include the distinct numbers $1, 2, r_1, r_2, 2r_1, 2r_2$. If, on the other hand, one of them, say $r_1$, is equal to 2, then the divisors include the distinct numbers $1, 2, 4, r_2, 2r_2, 4r_2$.

Case 2: If $t=r^n$ where $r$ is prime and $n \geq 2$, $r>2$, then the divisors of $p+q=2t$ include $1, 2, r, r^2, 2r, 2r^2$.

If $t=2^n$ then $p+q=2t$ is itself a power of 2. The first few pairs of consecutive primes give $p+q=5+7=12, 7+11=18, 11+13=24, \ldots$ and so we cannot have $p+q=4, 8$ or 16. It follows that the divisors of such $p+q=2t=2^{n+1}$ include the distinct numbers 1, 2, 4, 8, 16, 32.

In every case, we see, $p+q$ has at least 6 distinct positive divisors.

## Comments:

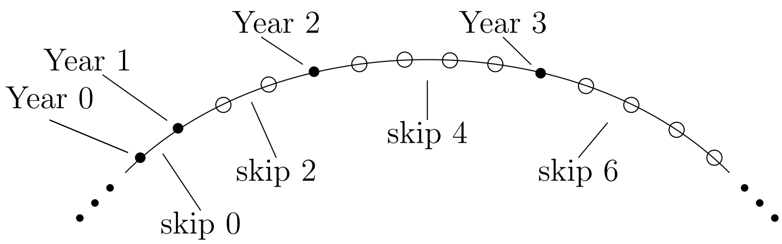
- Some of the casework is unnecessary in this solution. Incorporating $r_1<r_2$ in case 1 then the distinct divisors are given by $1, 2, 2r_1, r_2, 2r_2, 2r_1r_2$ without isolating the case $r_1=2$.

  More significantly, in Case 2 if $t=r^n$ where $r$ is prime, immediately $r=3$ since $p, 2t, q$ form an arithmetic progression, so one of them must be divisible by 3, and the desired divisors are $1, 2, 3, 6, 9, 18$.

- Many students misread the question or made fatal unwarranted assumptions. Commonly students didn't use the property of $p$ and $q$ being consecutive anywhere. One or two took this to mean that $q=p+1$ (i.e., that they are consecutive integers—obviously wrong). Others assumed incorrectly that there are (exactly) two cases, namely $q=p+2$ and $q=p+4$.

10. A secret society meets once per year in the middle of a circle of 101 equally spaced lanterns. In year 0 one lantern is lit. In year 1 they lit the adjacent lantern. After each year $k$, proceeding clockwise, they skip (leave untouched) $2k$ lanterns and change the state of the next lantern (so in year 2 they skip two lanterns, in year 3 they skip four lanterns, etc.—see diagram): If that lantern is lit, it would be extinguished; if unlit, it would be lit.

Determine, with proof, how many lanterns remain lit after they meet in year 2018.




**Solution:** Experimentation with the positions of the first few lanterns will show that, if we number them consecutively so the first-lit lantern is 0 and the second is 1 and so on, the candle visited in year $n$ is numbered $n^2$ (and when $n$ is large enough, it will be the remainder when $n^2$ is divided by 101).

We can show this by noting that the lantern affected is in position obtained by adding $2n-1$ to the position of the previous year's lantern. If in year $n$ the $n^2$ position is visited, in year $n+1$ the lantern in position $n^2+2(n+1)-1=(n+1)^2$ is visited.

Now in years $101k \pm n$ the position visited is the same as year $n$, since $(101 \pm n)^2$ differs from $n^2$ by a multiple of $101$.

When is the same position visited in two years, say $m$ and $n$? Equivalently, when does $101 | (m^2-n^2)$? We see that 101 is prime and $m^2-n^2=(m+n)(m-n)$ so this occurs when 101 divides one of $m+n$ or $m-n$. That is, $m=101k \pm n$ for some $k$.

So position 0 is visited in years that are a multiple of 101. Since $2018 = 19 \cdot 101 + 99$, that lantern is visited 19 times after it is lit in year 0, so it will be unlit after the 2017 gathering.

All other positions are either unvisited, or visited twice during each group of 101 years—once in years $101k+1, 101k+2, \ldots, 101k+50$ and then again in subsequent years $101k+51, \ldots, 101k+100$.

Thus each visited lantern other than the one in position 0 is visited 38 times (and so left unlit) prior to the final 99 years. Then 50 lanterns will be lit, then 49 extinguished, leaving exactly 1 lantern lit.

**Comments:**

- The the underlying machinery here is quadratic residues in modular arithmetic, but no knowledge of this is assumed
- Some messy details can be avoided with a slightly different approach. Let $l(y)$ be the lantern number that is changed in year y. After you get $l(y) = y^2 \pmod{101}$, note that $l(y) = l(y+101)$. So those two changes cancel each other. Similarly, all the changes in years 0–201 will cancel out, and same for 202–403, 404–605, ...,

1818–2019. So after year 2019, all lanterns are unlit. The only lantern lit after year 2018 is $l(2019)$. Final answer: one lantern.

- The main difficulty many students appeared to have is with reading comprehension. This is a relatively complex story problem in imprecise language and it includes a few distracting details. Students are expected to be prepared to cope with such complexity from time to time, and in this case we have a problem deliberately designed to separate top students from the pack. Careful reading and a strategy for verifying (say, from the examples shown in the diagram) that comprehension is correct should benefit students in this problem.
- Unfortunately, few students mounted more than a trivial attempt at this problem, and nobody solved it completely. One student came close, deriving that in every 202-year period every lantern will be put out once for every time it is lit, but failed to correctly account for what happens during the final 99 years.