2016 Manitoba Mathematical Competition Solutions

Thursday, February 24, 2016

1. (a) What is the sum of the digits in the number $N = 2^8 5^4$?

   (b) Find the largest three digit number divisible by both 5 and 6.

Solution:

(a) $2^8 5^4 = 2^4 (2 \cdot 5)^4 = 160000$, whose digits add to 7.

(b) Such a number must be a multiple of 30 (and vice versa). Since $30 \times 33 = 990$ and $30 \times 34 = 1020$ it follows that the required number is 990.

Notes:

- In part (a) some students worked out 2^8 and 5^4 separately and then multiplied—with mixed results.
- In part (b) many students tried an inefficient approach such as exhaustively listing the multiples of 30.

2. (a) A triangle initially has area 100 square units. If its base is increased by 10% and its altitude is decreased by 10% what is its new area?

   (b) What number is found two-thirds of the way from $\frac{1}{7}$ to $\frac{1}{5}$ on a number line?

Solution:

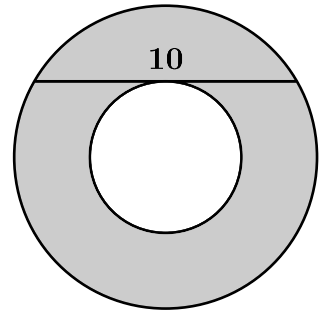
(a) With b for base length and h for height the initial area is $\frac{1}{2}bh = 100$ so the new area is $\frac{1}{2}(1.1)b \cdot (0.9)h = 0.99 \cdot \frac{1}{2}bh = 0.99 \cdot 100 = 99$ square units.

(b) $\frac{1}{7} + \frac{2}{3}\left(\frac{1}{5} - \frac{1}{7}\right) = \frac{1}{3} \cdot \frac{1}{7} + \frac{2}{3} \cdot \frac{1}{5} = \frac{5+14}{3 \cdot 5 \cdot 7} = \frac{19}{105}$.

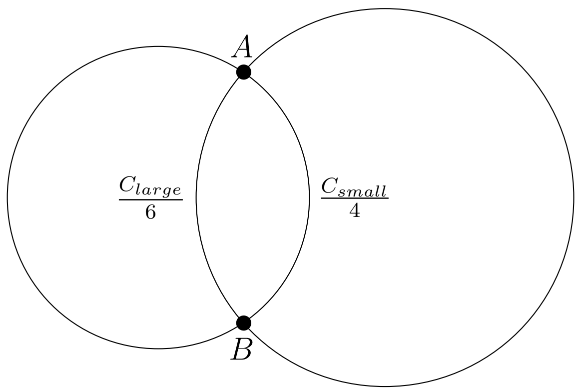
Notes:

- In (a) there was a tendency to work out the result with particular values of b and h. Strictly speaking this does not demonstrate one's answer to be correct, or rather assumes that the question does not depend on dimensions—not good form (but no marks lost for this).
- In (b) numerous students arrived at a final answer of $\frac{133}{735}$ which is correct, but not in lowest terms. Have we abandoned the important habit of reminding students to simplify one's work? In any case, this outcome only occurs when students fail to simplify the addition of fractions by finding a LCD. Because this inflates the size of numbers some students were getting lost in the calculations, wasting time, and losing marks.

3. (a) Two concentric circles have center O. A chord AB of the large circle tangent to the smaller circle has length 10. What is the area of the ring between the circles?

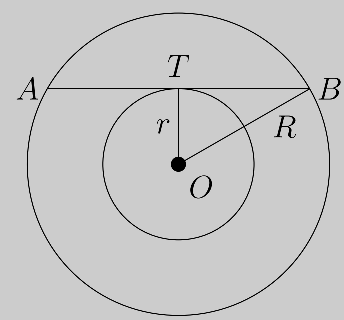


(b)



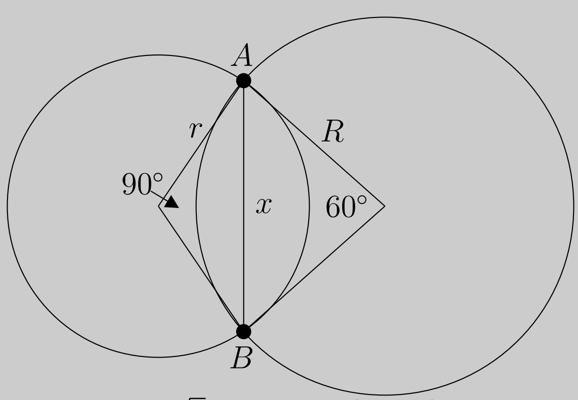
Two circles intersect at points A and B. The length of arc $\overset{\frown}{AB}$ on the smaller circle is $\frac{1}{4}$ of the circumference. The length of the corresponding arc on the larger circle is $\frac{1}{6}$ of its circumference. If area of the smaller circle is 12, what is the area of the larger circle?

Solution:



(a) Let the chord be AB, tangent at T, O the common center, the radii of the smaller and larger circles r and R respectively. Since $\angle OTB$ is a right angle, $5^2 = R^2 - r^2$. The shaded area is $\pi R^2 - \pi r^2 = \pi(R^2 - r^2) = \pi \cdot 5^2 = 25\pi$.

(b)



Let x be the length of the common chord AB. Using r and R for the radii of the smaller/larger circles respectively, we obtain $r = \frac{x}{\sqrt{2}}$ and $R = x$ (since AB forms right and 60° isosceles triangles at the respective centers). So $R : r = \sqrt{2} : 1$, so $\pi R^2 : \pi r^2 = 2 : 1$. Therefore the area of the larger circle is $2 \cdot 12 = 24$.

Notes:

- A “metasolution” to (a) drawing an assumption from the form of the question: “since the question does not give the radii of the two circles the answer must be the same for any setup satisfying the description. So let the smaller circle be degenerate, with radius 0—then AB is a diameter of the large circle and the required area is $\pi \cdot 5^2 = 25\pi$.”
- A further subtlety involved in such a solution, namely the judgement that information is genuinely missing–a proper execution would have to first demonstrate that one cannot determine the radii from the given information, which seems intuitively clear but without such a demonstration, the reduction to the case in which one circle is degenerate is unwarranted. The question is academic: no students tried this but it may have been worth some partial marks.

4. (a) The sum of two numbers is 10 and their product is 20. What is the sum of their cubes?

   (b) Can one make the expression $1 \star 2 \star 3 \star \cdots \star 10$ equal to 0 by replacing each instance of "$\star$" with either "+" or "−"?

Solution:

(a) Let the numbers be a and b. We are given that $a + b = 10$, $ab = 20$. Thus

$$10^2 = (a+b)^2 = a^2 + b^2 + 2ab = a^2 + b^2 + 40$$

so $a^2 + b^2 = 60$. Now,

$$a^3 + b^3 = (a+b)(a^2 - ab + b^2) = 10(60 - 20) = 400$$

Alternative (Viète's relations)

a, b are roots of the polynomial $t^2 - 10t + 20$, so both satisfy $t^2 = 10t - 20$, so also $t^3 = 10t^2 - 20t = 10(10t - 20) - 20t = 80t - 200$. So $a^3 + b^3 = 80(a+b) - 2 \cdot 200 = 800 - 400 = 400$.

Alternative (one-line arrangement)

$$a^3 + b^3 = (a+b)^3 - 3ab(a+b) = 10^3 - 3 \cdot 20 \cdot 10 = 1000 - 600 = 400.$$

(b) Any expression thus obtained will be equal to $1 + 2 + \cdots + 10 - 2S = 55 - 2S$ where S is the sum of all subtracted terms. Since 55 is odd $55 - 2S$ is also odd and so is nonzero. So the answer is "no".

Notes:

- In (a) one could also brute force the answer by determining that $\{a, b\} = \{5 \pm \sqrt{5}\}$ so $a^3 + b^3 = (5 + \sqrt{5})^3 + (5 - \sqrt{5})^3$. Every second term in the two expansions cancel, giving $a^3 + b^3 = 2\left(5^3 + 3 \cdot 5 \cdot \sqrt{5}^2\right) = 400$. Correctly done this is worth full marks, however awkward it may be.
- Numerous students showed difficulty interpreting the intention of (b). The question is not ambiguous—the problem for most such students appeared to arise because of reading the word "each" as "all".

5. Define an operation, $*$, for positive real numbers as follows:

$$a * b = \frac{ab}{a+b}.$$

(a) Verify that $(2 * 2) * 3$ is equal to $2 * (2 * 3)$.

(b) Is it always the case that $a * (b * c) = (a * b) * c$? Justify your answer.

Solution:

(a) Direct calculation in both cases gives $\frac{3}{4}$.

(b) $a * (b * c) = a * \frac{bc}{b+c} = \frac{\frac{abc}{b+c}}{a+\frac{bc}{b+c}} = \frac{abc}{ab+bc+ca}$. Either direct calculation shows the same for $(a * b) * c$ or one can simply observe that the final expression is symmetric in a, b, c. So $a * (b * c) = c * (a * b)$ and the obvious commutativity of $*$ gives $\cdots = (a * b) * c$.

Alternative (b) Solution

$\frac{1}{a*b} = \frac{a+b}{ab} = \frac{1}{a} + \frac{1}{b}$. So $\frac{1}{(a*b)*c} = \frac{1}{a*b} + \frac{1}{c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

Notes:

- In the contest paper we inadvertently left the mark breakdown (omitted here—such annotations are for discussion as the contest paper goes through draft discussions). For anyone interested, the 3-7 allocation of marks seen on the paper for the two parts was indeed applied.
- Observe that $(b) \Rightarrow (a)$. Solving the problem in that order shows good tactical sense.; full credit was given for this approach if explicitly explained.
- A lovely extension for this question: explore what goes wrong when $*$ is defined over the real numbers instead of the positive reals. Consider what happens when $a + b = 0$. (b) would make sense only for those values of a, b, c for which *both expressions are defined*. Explore what goes wrong when $a + b = 0$ or $b + c = 0$! Also the case $ab+bc+ac = 0$. And here's a lovely paradox for your best students: Is it possible for both expressions to be defined, in general, when $a + c = 0$ as long as $a + b, b + c$ and $ab + bc + ac \neq 0$? Try it with, say, $(a, b, c) = (1, 2, -1)$. So the associative law applies in this case? Now clearly $*$ is commutative. So in this case we must have $a * (b * c) = a * (c * b) = (a * c) * b$? Try it!

6. How many different positive integers n have the property that $n^2 - 2016$ is a perfect square? Find the smallest such number.

Solution: Suppose $n \in \mathbb{Z}$ such that $n^2 - 2016 = k^2$, $k \in \mathbb{Z}^+$. Then

$$n^2 - k^2 = (n+k)(n-k) = 2016 = 2^5 \cdot 3^2 \cdot 7$$

breaks 2016 into factors, $n \pm k$, of equal parity. Since 2016 is even both factors must also be even. Further each such factorization yields a pair n, k. Writing $n + k = 2^a 3^b 7^c$ with $a \in \{1, 2, 3, 4\}$, $b \in \{0, 1, 2\}$ and $c \in \{0, 1\}$ provides $4 \cdot 3 \cdot 2 = 24$ cases, 12 of which have $n + k > n - k$ (i.e, $k > 0$), so there are 12 possible values of n.
Fixing $N = xy$ the value $x + y = x + \frac{N}{x}$ is minimized when $x = y = \sqrt{N}$ and increases as a function of $|x - \sqrt{N}|$ on interval $[0, \sqrt{N}]$ and also on $[\sqrt{N}, N]$.
So $n = \frac{(n+a)+(n-a)}{2}$ is minimized by taking $n + a = 2^a 3^b 7^c$ as close as possible above $\sqrt{2016} = 12\sqrt{7} \in \{44, 45\}$.
This yields $n + a = 48$, giving minimum value $n = \frac{42+48}{2} = 45$.

Notes:

- Some tried to list all 24 even divisors of 2016 (the student wastes time on such approaches, risking costly elementary errors, at their own peril).
- Justification of minimality is tricky here. Many approaches tried.

7. How many of the first 2016 positive integers are divisible by none of 7, 9 and 32?

Solution: Here $2016 = 32 \cdot 7 \cdot 9$. If we exclude $\frac{2016}{32} = 63$ multiples of 32; $\frac{2016}{7} = 288$ multiples of 7; and $\frac{2016}{9} = 224$ multiples of 9, then we have double-counted multiples of $7 \cdot 32$, $7 \cdot 9$ and $9 \cdot 32$—and triple-counted the one multiple of all three numbers, 2016 itself. Subtracting these (and adding 1 for the triple-counted number, which is subtracted 3 times without this) and subtracting the result from 2016 the answer is

$$2016 - (63 + 288 + 224) + (32 + 7 + 9) - 1 = 1488$$

Notes:

- We have described inclusion/exclusion in verbal form. Use of a Venn Diagram is also acceptable—even a solution that involves labelling regions of a Venn diagram with little explanation, as long as the diagram makes the chain of reasoning clear.

8. Let $a_k = \frac{1}{\sqrt{2k-1}+\sqrt{2k+1}}$. Determine n so that $a_1 + a_2 + a_3 + \cdots + a_n = 8$.

Solution: $a_k = \frac{1}{\sqrt{2k-1}+\sqrt{2k+1}} \cdot \frac{\sqrt{2k-1}-\sqrt{2k+1}}{\sqrt{2k-1}-\sqrt{2k+1}} = \frac{\sqrt{2k-1}-\sqrt{2k+1}}{(2k+1)-(2k-1)} = \frac{\sqrt{2k+1}-\sqrt{2k-1}}{2}$. Now we have

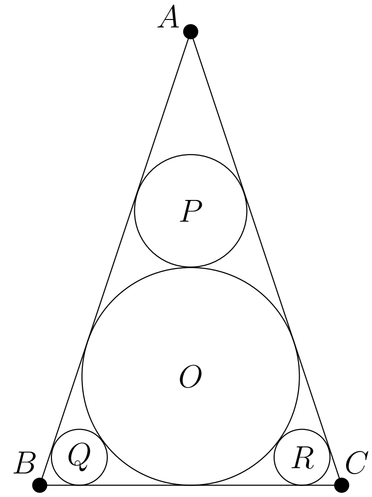
$$\begin{aligned} a_1 + a_2 + \cdots + a_n &= \frac{\sqrt{3}-\sqrt{1}}{2} + \frac{\sqrt{5}-\sqrt{3}}{2} + \frac{\sqrt{7}-\sqrt{5}}{2} + \cdots + \frac{\sqrt{2n+1}-\sqrt{2n-1}}{2} \\ &= \frac{\sqrt{3}-\sqrt{1}+\sqrt{5}-\sqrt{3}+\sqrt{7}-\sqrt{5}+\cdots+\sqrt{2n+1}-\sqrt{2n-1}}{2} \\ &= \frac{-1+\sqrt{2n+1}}{2} = 8 \end{aligned}$$

So $\sqrt{2n+1} = 17$, and $n = \frac{17^2-1}{2} = 144$.

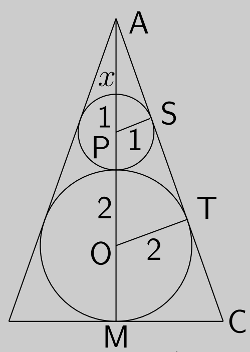
Notes:

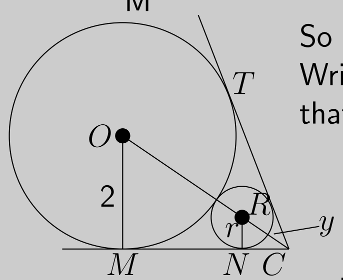
- It is expected that some version of a telescoping sum is used to simplify.
- Some students worked out a_n for a few small values of n, guessed a pattern and used the resulting formula. This is incorrect form and loses marks. What is required is a deductive procedure: mathematics is not guesswork. An inductive approach may be helpful for exploration but does not adequately justify such a step (in a full-solution contest every solution ought to be constructed so as to read as a proof that one's answer is correct). Students need to understand the distinction between guessing an answer and demonstrating that it is as claimed.

9. Circle O is inscribed (as in the diagram) in isosceles triangle $\triangle ABC$ where $AB = AC$. Circles P, Q and R are each tangent to two sides of this triangle and, externally, to O as shown. The radius of O is 2 units and the radius of P is 1 unit. Find the radii of the other two circles.



Solution:





We consider the larger circles P and O, also denoting their centers by these labels. $\triangle APS \sim \triangle AOT$, so $\frac{AP}{AO} = \frac{PS}{OT}$. With x as in the diagram this gives $\frac{x+1}{x+4} = \frac{1}{2}$, which gives $x = 2$. So $AP = 3$ and $AS = 2\sqrt{2}$ by Pythagoras' theorem. Also $\triangle APS \sim \triangle AMC$, so $\frac{MC}{PS} = \frac{AM}{AS}$, thus $\frac{MC}{1} = \frac{3+1+2+2}{2\sqrt{2}}$ which gives $MC = 2\sqrt{2}$.

Taking r to be the common radius of circles Q and R, we similarly take y to be the distance from C to where the segment CO strikes circle R. Applying Pythagoras' Theorem to $\triangle OMC$ we obtain that $CO = \sqrt{(2\sqrt{2})^2 + 2^2} = 2\sqrt{3}$. So $y + 2r + 2 = 2\sqrt{3}$ (**).

Writing N for the point where circle R meets BC, we have that $\triangle CRN \sim \triangle COM$, whence we have

$$\frac{r+y}{r} = \frac{2\sqrt{3}}{2} = \sqrt{3}$$

and so $y = r(\sqrt{3} - 1)$.

Substituting into (**) and simplifying gives $r = 4 - 2\sqrt{3}$.

Notes:

- Brute force plus standard geometric technique, but there are enough steps to throw off or trip up all but the most determined solver. For students who found themselves mired in the steps of this problem it may be comforting to know that our original key contained a major error that was only caught on marking day.

10. Let x, y and z be real numbers satisfying $x+y+z=15$ and $xy+yz+zx=72$. Prove that $3 \leq x \leq 7$.

Solution:

$$x^2+y^2+z^2=(x+y+z)^2-2(xy+yz+xz)=15^2-2(72)=8.$$

Therefore $y+z=15-x$ and $y^2+z^2=81-x^2$. Let $k=15-x$ and $m=81-x^2$ so that $y+z=k$ and $y^2+z^2=m$.
So we have

$$m=y^2+(k-y)^2=2y^2-2ky+k^2$$

Whence the quadratic $2y^2-2ky+(k^2-m)=0$, whose discriminant$(2k)^2-4\cdot 2(k^2-m)$ must be ≥ 0 for a real solution to exist. Therefore $2m-k^2\geq 0$. Substituting back and simplifying gives $-3x^2+30x-63\geq 0$. Dividing by -3 and factoring: $(x-3)(x-7)\leq 0$. It follows that $3\leq x\leq 7$.

Alternative #1

Write $y+z=15-x$ and $yz=72-x(y+z)=72-x(15-x)=x^2-15x+72$. Regarding x temporarily as a constant, y and z are roots of the quadratic equation

$$t^2-(15-x)t+(x^2-15x+72)=0$$

whose discriminant is $(15-x)^2-4(x^2-15x+72)=-3(x^2-10x+21)$.
Since y, z are to be real this must be positive, so $(x-3)(y-7)\leq 0$, as above.

Alternative #2 (geometric)

The given system of equations is equivalent to $x+y+z=15$ and $x^2+y^2+z^2=81$, the equations of a sphere and a plane. We seek bounds on the x value on the circle of intersection. Since a normal to the plane through $O(0,0,0)$ passes through the centre of the circle, that centre is $M(5,5,5)$. Since $(6,6,3)$ is on that circle, its radius is $\sqrt{(6-5)^2+(6-5)^2+(3-5)^2}=\sqrt{6}$.
The plane cuts off a tetrahedron in the first octant. Three sides of the tetrahedron are right isosceles triangles with legs of 15, and the third side is an equilateral triangle with sides $5\sqrt{2}$. The centre, M, of the circle is also the centroid of the equilateral triangle. If A is the x-intercept of the plane, then $\triangle OMA$ is right angled at M. All three sides are easily determined; OA is the x-axis. By similar triangles, a distance of $\sqrt{6}$ along MA projects to give a distance of 2 along the x-axis. Thus the values of x range from $5-2$ to $5+2$.

****...Continued...****

Solution:

Alternative #3 (Viète's relations)

x, y, z are the zeros of polynomial $f(t) = t^3 - 15t^2 + 72t - C$, where $C = xyz$. Since $f'(t) = 3(t-4)(t-6)$, $f(t)$ has three real roots only if C is chosen such that $f(6) \leq 0 \leq f(4)$, with minimum possible root (as evident from a sketch of the curve) as the third root when there is a double root at $t = 6$. $f(6) = 216 - 540 + 432 - C = 108 - C$, so we take $C = 108$ giving $f(t) = (t-6)^2(t-3)$ and the minimum value of x is therefore 3. Similarly $f(4) = 64 - 240 + 288 - C = 112$; taking $C = 112$ gives $f(x) = (t-4)^2(t-7)$ so the maximum value of x is 7.

Alternative #4

Write $y + z = 15 - x$ and $yz = 72 - x(y+z) = 72 - x(15-x) = x^2 - 15x + 72$. Then

$$\begin{aligned} 0 &\leq (y-z)^2 = (y+z)^2 - 4yz = (15-x)^2 - 4(x^2 - 15x + 72) \\ &= 225 - 30x + x^2 - 4x^2 + 60x - 288 = -3x^2 + 30x - 63 = -3(x-3)(x-7) \end{aligned}$$

The result follows.

Notes:

- Here is a flawed but tantalizingly approach (we cannot assume from the start that $x, y, z \geq 0$): By AM/GM (with that assumption...) $(15-x)/2 = (y+z)/2 \geq \sqrt{yz} = \sqrt{72 - x(y+z)} = \sqrt{72 - x(15-x)} = \sqrt{x^2 - 15x + 72}$, whence $(x-3)(x-7) \leq 0$ as above.
- Preconditioning the system as follows simplifies both algebraic solutions: the average of x, y, and z is $\frac{x+y+x}{3} = \frac{15}{3} = 5$. Let $x = 5 + r$, $y = 5 + s$, $z = 5 + t$. Then $r + s + t = 0$ and $rs + st + tr = -3$. We end up with $-2 < r < 2$.