2008 MANITOBA MATHEMATICAL COMPETITION
SOLUTIONS

1. (a) Solve the equation $x + \frac{6}{x+1} = 4$.
   (b) Solve the equation $x^5 + 36x = 13x^3$.

Solution: *(a) Multiply both sides by $x + 1$:*

$$x(x+1) + 6 = 4(x+1), \qquad x \neq -1$$
$$x^2 - 3x + 2 = (x-1)(x-2) = 0.$$

Thus $x = 1, 2$ (both validated by substituting into original equation).

(b) Gather terms on left, simplify and factor:

$$x^5 - 13x^3 + 36x \;=\; x(x^4 - 13x^2 + 36) \;=\; x(x^2-4)(x^2-9) \;=\; 0.$$

Thus $x = 0, \pm 2, \pm 3$ (solutions validated by substituting into original equation)

Note: In both parts the validation step is unnecessary because the solution is extracted from a relation clearly equivalent to the original. (In (a), $x = -1$ is not a possible solution.)

2. (a) Find the real numbers a and b if 2 and 3 are roots of $x^3 + ax^2 + bx + 6 = 0$.
   (b) In this problem A and B are the two points at which the graph of the equation $x^2 + y^2 = 8$ meets the graph of the equation $y = |x|$. What is the length of the segment AB?

Solution: *(a) Substitute 2 and 3 into the equation:*

$$8 + 4a + 2b + 6 = 0$$
$$27 + 9a + 3b + 6 = 0.$$

Simplify:

$$2a + b = -7 \qquad (1)$$
$$3a + b = -11. \qquad (2)$$

Subtract (2)−(1) to obtain $a = -4$. Substituting into (1) gives $(a, b) = (-4, 1)$.

Alternate: *(a) The product of the roots is -6 so the third root is -1. Thus the polynomial is $(x-2)(x-3)(x+1) = x^3 - 4x^2 + x + 6$, and $(a, b) = (-4, 1)$.*

(b) By the second condition $y^2 = x^2$, so $2x^2 = 8$; $x = \pm 2$, $y = 2$. The points A and B are $(-2, 2)$ and $(2, 2)$, so the length of segment AB is $2 - (-2) = 4$.

3. (a) Find an equation of the circle passing through the origin and the points with coordinates $(10, 0)$ and $(0, 8)$.

   (b) Find an equation of the line tangent to the circle with equation $(x-2)^2 + (y+1)^2 = 25$ at the point with coordinates $(5, 3)$.

Solution: *(a) Since $A(0,0)$, $B(10,0)$ and $C(0,8)$ are vertices of a right triangle the center of the circle is the midpoint, $D(5,4)$, of the hypotenuse BC and its radius is $|DC| = \sqrt{41}$. The required equation is therefore $(x-5)^2 + (y-4)^2 = 41$.*

(b) The slope of the radius from the center $(2,-1)$ to the point of tangency $(5,3)$ is $\frac{3+1}{5-2} = \frac{4}{3}$; the slope of the tangent line is thus $-\left(\frac{4}{3}\right)^{-1} = -\frac{3}{4}$. The required equation, in point-slope form, is therefore $y - 3 = -\frac{3}{4}(x-5)$.

Alternate answers: *(b) Slope-intercept form: $y = -\frac{3}{4}x + \frac{27}{4}$*
Standard form: $3x + 4y = 27$.

4. (a) In this problem c and d are real numbers. The point on the graph of the equation $y = x^2 + cx + d$ which is nearest to the x-axis is $(-2, 5)$. find the values of c and d.

   (b) Car A is travelling due west at a constant speed of 50 km/hr. Car B is travelling due east at a constant speed of 60 km/hr. At 1:00 p.m. car A is 40 km due north of car B. At 2:00 p.m. what is the distance between the two cars (as the crow flies)?

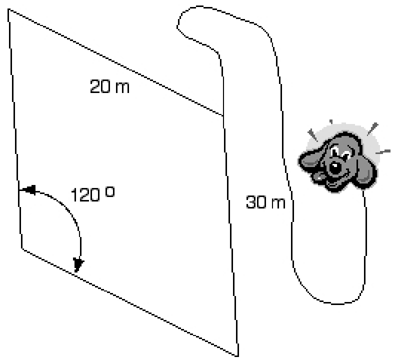
Solution: *(a) The equation is that of a vertically oriented parabola, and may be rewritten*

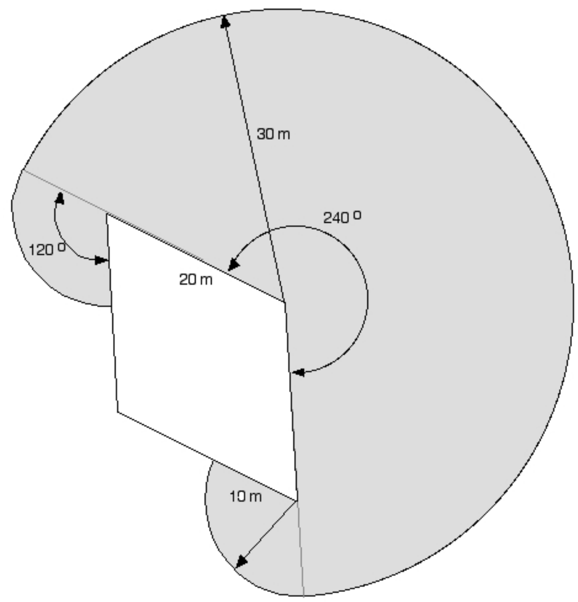
$$y = \left(x + \frac{c}{2}\right)^2 + d - \frac{c^2}{4}.$$

The point nearest the axis is clearly the vertex, so $\left(-\frac{c}{2}, d - \frac{c^2}{4}\right) = (-2, 5)$. The two corresponding equations immediately yield $(c, d) = (4, 9)$.

(b) In a standard coordinate system with the usual orientation, miles for units and B at the origin at 1:00 p.m, the coordinates of the cars at 2:00 p.m. are $A(-50, 40)$ and $B(60, 0)$. From the distance formula (or Pythagoras: hypotenuse of a right triangle with sides 40 and 110) the required distance is $\sqrt{40^2 + (50+60)^2} = 10\sqrt{137}$ km.
(Approximate value ≈ 117.047 km. — not required.)

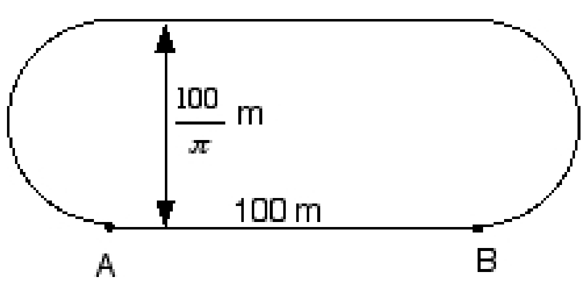
5. A fenced property has the shape of a rhombus, as in the figure. The length of each side of the rhombus is 20 m. A dog outside the property is tethered to one corner of the rhombus as shown in the diagram. If the dog's leash is 30 m long, how large an area can the dog cover?





Solution: *Sketching the boundary one obtains that the region described consists of $\frac{2}{3}$ of a circle of radius 30 m and $\frac{2}{3}$ of another circle radius 10 m (in two sectors each subtending $120°$). Thus the total area is $\frac{2\pi\cdot 30^2}{3} + \frac{2\pi\cdot 10^2}{3} = \frac{2000\pi}{3}$ m^2.*

6. A race track is built with two straight parallel sides and semicircles at the ends (as in the figure). The parallel sides are 100 m long and $\frac{100}{\pi}$ m apart. Runner Alpha at position A starts running clockwise around the track at 2 m/sec. At this precise moment a second runner Beta enters the track at position B which is 100 m from position A, running at 5 m/sec. If Beta wants to meet Alpha as soon as possible, should he run clockwise or counterclockwise around the track to achieve his goal?



Solution: *The semicircles together equal the circumference of a circle of radius $\frac{50}{\pi}$, or 100 m.* **Running clockwise:** *the distance to cover is 100 m at an effective speed of $5-2=3$ m/sec, which will take $\frac{100}{3} = \frac{200}{6}$ sec.* **Running counterclockwise:** *the distance to cover is $100+100=200$ m at an effective speed of $5+2=7$ m/sec, which will take $\frac{200}{7} < \frac{200}{6}$ sec. Beta should run counterclockwise to achieve his goal.*

7. For what values of x does $\frac{1}{x+1} + \frac{1}{2x} > 1$ hold?

Solution: *Let* $f(x) = \frac{1}{x+1} + \frac{1}{2x} - 1 = \frac{-2x^2+x+1}{2x(x+1)} = \frac{(2x+1)(1-x)}{2x(x+1)}$. *Sign analysis:*

interval:	$(-\infty,-1)$	$(-1,-\frac{1}{2})$	$(-\frac{1}{2},0)$	$(0,1)$	$(1,\infty)$
$2x+1$	$-$	$-$	$+$	$+$	$+$
$1-x$	$+$	$+$	$+$	$+$	$-$
$2x$	$-$	$-$	$-$	$+$	$+$
$x+1$	$-$	$+$	$+$	$+$	$+$
$f(x)$	$-$	$+$	$-$	$+$	$-$

The inequality holds where $f(x) > 0$*—that is, for* $-1 < x < -\frac{1}{2}$ *and* $0 < x < 1$.

Alternate analysis: $f(-2) < 0$ and the value of f changes sign at $-1, -\frac{1}{2}, 0, 1$ (because these are poles/roots of odd degree); the solution is immediate.

8. In this problem x, y and z are real numbers. Find all possible values of a if:

$$a = \frac{x}{|x|} + \frac{y}{|y|} + \frac{z}{|z|}.$$

Solution: *Each term is* ± 1*, according as the variable involved is* > 0 *or* < 0*. Since the variables are independent, the possible sums are any number of the form* $a = p+q+r$*, where* $p, q, r \in \{\pm 1\}$*. That is,* $a \in \{-3, -1, 1, 3\}$.

9. Prove that, if $a + b + c = 0$, then $a^3 + b^3 + c^3 = 3abc$.

Solution: *Form the polynomial with roots* a, b, c*:* $p(x) = x^3 - (a+b+c)x^2 + (ab+ac+bc)x - abc$*. Directly, we have*

$$\begin{aligned} p(a)+p(b)+p(c) &= a^3+b^3+c^3 - (a+b+c)(a^2+b^2+c^2) + (ab+ac+bc)(a+b+c) - 3abc \\ &= a^3+b^3+c^3 - 3abc = 0+0+0 = 0, \end{aligned}$$

and the conclusion follows immediately.

Alternate #1: *Cube* $a+b+c$*:*

$$\begin{aligned} 0 &= (a+b+c)^3 \\ &= a^3+b^3+c^3+3ba^2+3ca^2+3ab^2+3cb^2+3ac^2+3bc^2+6abc \\ &= a^3+b^3+c^3+3ab(a+b)+3ac(a+c)+3bc(b+c)+6abc \\ &= a^3+b^3+c^3-3ab(c)-3ac(b)-3bc(a)+6abc = a^3+b^3+c^3-3abc, \end{aligned}$$

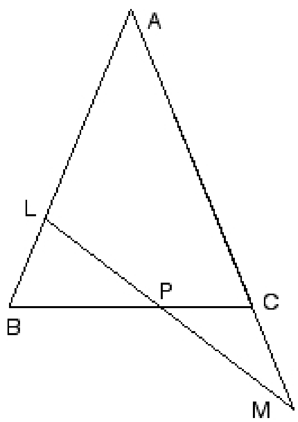
and the result follows immediately.

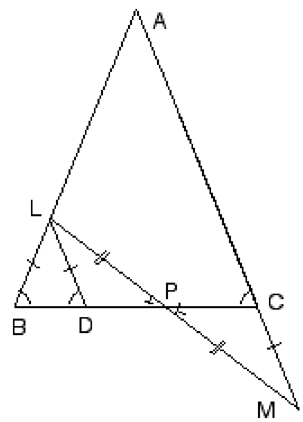
Alternate #2: *$c = -(a+b)$, so*

$$\begin{aligned} a^3+b^3+c^3 &= a^3+b^3-(a+b)^3 \\ &= (a+b)(a^2-ab+b^2-(a+b)^2) \\ &= -c(-3ab) = 3abc \end{aligned}$$

and the result follows immediately.

10. In the diagram $\triangle ABC$ is isosceles with $AB = AC$. Prove that if $LP = PM$, then $LB = CM$.





Solution: *Add point D on BC so that $LD||AC$. Then $\angle BDL = \angle PCA$, $\angle LDP = 180 - \angle BDL = 180 - \angle PCA = \angle PCM$, and $\angle CPM = \angle DPL$, so $\triangle DLP \cong \triangle CMP$, by SAA. Further, $\triangle BDL$ is isosceles with base BD, so $LB = LD = CM$, as required.*

Alternate solution: *As in the first solution, $\angle BPL = \angle CPM$, while $\angle PBA$ and $\angle PCM$ are supplementary. Since supplementary angles have the same sine, by the sine law we have*

$$\frac{|LB|}{\sin \angle BPL} = \frac{|LP|}{\sin \angle PBA} = \frac{|PM|}{\sin \angle PCA} = \frac{|PM|}{\sin \angle PCM} = \frac{|CM|}{\angle CPM} = \frac{|CM|}{\sin \angle BPL},$$

and the result follows immediately.