2018 MANITOBA MATHEMATICAL COMPETITION

(for students in grade 12)

Tuesday, February 27, 2018, 9 AM – 11 AM

Sponsors:

MAMT Manitoba Association of Mathematics Teachers

Winnipeg Actuaries' Club

University of Manitoba

Canadian Mathematical Society

☐ Mr.
☐ Mrs.
☐ Ms.

↑ Student's Name (print). Underline the Surname ↑

↑ Student's Signature ↑

↑ Student's School ↑

↑ Student's Home Street Address (or P. O. Box #) ↑

↑ City/Town Postal Code ↑

Instructions for participants: Before the contest begins, complete the above information. Put no personal identifying information on any other pages. You should have received 13 pages in total, including this page.

Answer each question on the page where it appears. You may use both front and back of the page. The last two QR-coded pages are provided for scratch calculations or for continuing solutions where you run out of space. Work continued on these pages will not be credited unless where continuation occurs a page location for it is clearly specified, and any continuation clearly indicates which question is being continued.

In your work on any question do not refer to work done on any other questions; they are marked independently.

No aids are permitted—no straight edges, compasses or other mechanical drawing devices, electronics (cell phones, electronic watches, translators, tablets, calculators etc).

This space may be used for scratch work, but do not continue solutions on this page. No credit will be given for work appearing here.

Question 1

(a) Find an integer greater than 1 that leaves remainder 1 when divided by each of 2, 3, 4, 5 and 6.

(b) If $a^2 - b^2 = 42$ and $2a + 2b = 14$, find $3a - 3b$.

Question 2

(a) Solve for x:

$$x\sqrt{x+5} + 5\sqrt{x+5} \;=\; 8$$

(b) Solve for a and b:

$$\begin{aligned} a + b &= 53 \\ \sqrt{a} - \sqrt{b} &= 5 \end{aligned}$$

Question 3

(a) What is the sum of the digits of $10^{50} - 5^5$?

(b) Evaluate the sum $f(1) + f(2) + f(3) + \cdots + f(100)$, where $f(k) = \dfrac{1}{4k^2 - 1}$.

Question 4

(a) Determine the sum of all five-digit numbers whose digits consist only of 2s or 3s.

(b) A sequence of real numbers is defined by $x_0 = 1$ and $x_{k+1} = \dfrac{2x_k - 4}{x_k + 2}$ for all $k \geq 0$. Determine the value of x_{2018}.

Question 5

(a) For how many positive integers n is $\frac{n}{3}$ a three-digit integer and $6n$ a four-digit integer?

(b) Suppose a, b, c are nonzero numbers and $ax + a = by + b = cz + c = a + b + c$.
Use this information to find the numerical value of the expression $xyz - (x + y + z)$.

Question 6

(a) Prove that, in any set $\{a_1, a_2, a_3, a_4\}$ of four integers, there are two elements that differ by a multiple of 3.

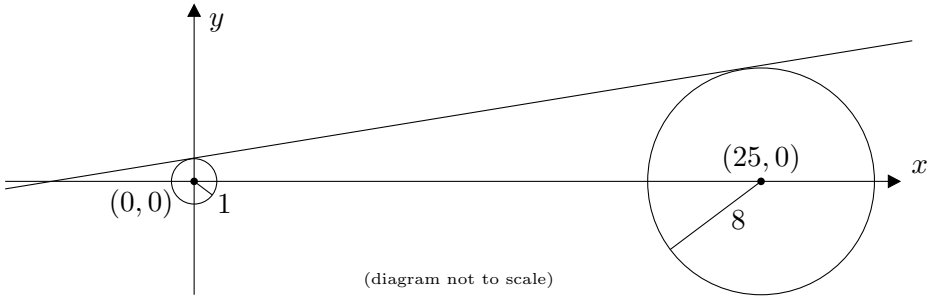
(b) Prove that no positive integer has a square of the form $100k + 51$

Question 7

The line $2x + y - 12 = 0$ intersects the parabola $y = x^2 - 4x + 9$ at points A and B.
If C is the vertex of the parabola, find the area of $\triangle ABC$.

Question 8

A circle with radius 1 has centre at $(0, 0)$. A second circle with radius 8 has centre at $(25, 0)$. A line lies above both circles and is tangent to both, as illustrated. Find an equation for this line.

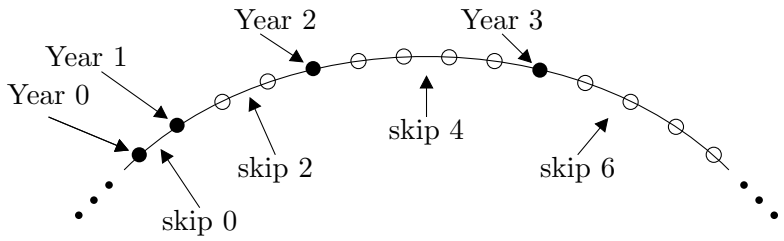


Question 9

If p and q are consecutive prime numbers, both greater than 3, prove that $p+q$ has at least 6 distinct positive divisors. (For example, $31 + 37 = 68$ which is divisible by $1, 2, 4, 17, 34$ and 68.)

Question 10

A secret society meets once per year in the middle of a circle of 101 equally spaced lanterns. In year 0 one lantern is lit. In year 1 they lit the adjacent lantern. After each year k, proceeding clockwise, they skip (leave untouched) $2k$ lanterns and change the state of the next lantern (so in year 2 they skip two lanterns, in year 3 they skip four lanterns, etc.—see diagram): if that lantern is lit, it would be extinguished; if unlit, it would be lit.
Determine, with proof, how many lanterns remain lit after they meet in year 2018.



This page (front and back) is for scratch work or continuation of solutions.

To receive credit for work continued here:

1. Clearly indicate in your solution that it is continued and exactly where
2. Where it is continued, clearly indicate which question is being continued
3. Do not mix up continued work from different questions or with scrap work

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