2015 MANITOBA MATHEMATICAL CONTEST

For students in grade 12
9:00 AM – 11:00 AM
Tuesday, February 24, 2015

Manitoba Association of Mathematics Teachers

Winnipeg Actuaries' Club

Sponsored by:

The Winnipeg Actuaries' Club

The Manitoba Association of Mathematics Teachers

The Canadian Mathematical Society

The University of Manitoba

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Questions are found on both sides of this sheet. Answer as many as possible, but you are not expected to answer them all. **CALCULATORS ARE NOT PERMITTED**. Numerical answers by themselves, without explanation, will not receive full credit.

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1. (a) Solve for x:

$$\frac{5}{x} - \frac{7+2x}{3x} = 3$$

(b) Find the ratio of x to y, given that

$$\frac{x-y}{x+y} = \frac{2}{5}$$

2. (a) Let "$*$" denote an operation defined as follows: $a * b = 2ab - b^3$.
Find the value of $(4 * 3) * 2$.

(b) The original price of an item is reduced by 20%. A month later its reduced price is doubled. At the end of the year it is sold in a discount sale. What exact percentage discount was applied if the sale price was equal to the original price?

3. In a **Fibonacci-like sequence** each term is the sum of the two terms immediately preceding it. For example the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ and the Lucas sequence $1, 3, 4, 7, 11, 18, \ldots$ are both Fibonacci-like sequences.

(a) The first term of a Fibonacci-like sequence is 4 and the fifth term is 2. Find the second, third and fourth terms.

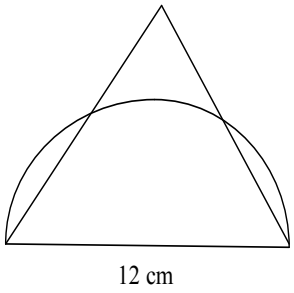
(b) If $x, y, 2x - 1, x + 4$ is a Fibonacci-like sequence, find the values of x and y.

4. (a) Find, with justification, all pairs of real numbers (a, b) such that $a^2 + 2ab + b^2 = 9$ and $a - b = 5$.

   (b) Find, with justification, all pairs of real numbers (a, b) such that $a^2 - ab + b^2 = 0$.

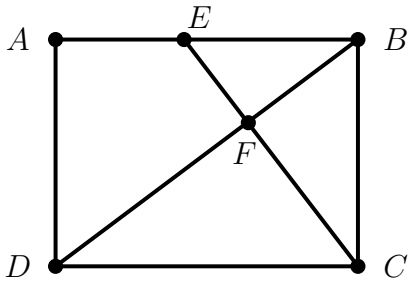
5. (a) Find all possible values of a such that the line with equation $x + y = a$ is tangent to the circle with equation $x^2 + y^2 = 25$.

   (b) One side of an equilateral triangle is the 12 cm diameter of a semi-circle, as shown. Find the area of the portion of the triangle that is outside the semi-circle.



6. The sides of a right triangle are $x + 2$, $x + 6$ and $2x$. Find all possible values of x.

7. $ABCD$ is a rectangle. E is on AB. CE intersects diagonal DB at F. $\triangle ADE$ has area 50. $\triangle EFB$ has area 40. Find the area of rectangle $ABCD$.



8. The lines $y = -2x + 12$, $x = 2$ and $y = mx$ intersect to form a triangle with area 2. Find all possible values for m.

9. Let A, B, C be vertices of an equilateral triangle, with D an interior point such that the lengths of AD, BD, CD are 3, 4 and 5, respectively. What is the area of the triangle?

10. The expression $n!$ denotes the product $1 \cdot 2 \cdot 3 \cdots n$ and is read as "n factorial".
    For example $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

    (a) The product $(2!)(3!)(4!)(5!)(6!)(7!)(8!)(9!)(10!)(11!)(12!)$ can be written in the form $M^2N!$ where M, N are positive integers. Find a suitable value of N and justify your answer.

    (b) Prove that, for every $n \geq 1$, $(2!)(3!)(4!) \cdots ((4n)!)$ can be written as the product of a square and a factorial.