MANITOBA MATHEMATICAL CONTEST, 2004

For Students in Senior 4

9:00 AM – 11:00 AM

Wednesday, February 18, 2004

Sponsored by

The Actuaries' Club of Winnipeg

The Manitoba Association of Mathematics Teachers

The Canadian Mathematical Society

and

The University of Manitoba

ANSWER AS MUCH AS POSSIBLE. YOU ARE NOT EXPECTED TO COMPLETE THE PAPER. SEE BOTH SIDES OF THIS SHEET. HAND CALCULATORS ARE NOT PERMITTED. NUMERICAL ANSWERS ONLY, WITHOUT EXPLANATION, WILL NOT BE GIVEN FULL CREDIT.

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1. (a) If a and b are real numbers such that $a^3b + ab^3 = 24$ and $ab = 2$ find the numerical value of $a^2 + b^2$.

   (b) If c and d are real numbers such that $\dfrac{c - 3d}{c + 7d} = \dfrac{2}{5}$ find the numerical value of $\dfrac{c}{d}$.

2. (a) Find the area of a circle which passes through the four vertices of a square whose area is 12.

   (b) In this problem O is the origin, A and B are points on the graph of the equation $y = x^2$ and AB is parallel to the x-axis. If the area of triangle AOB is 8 what is the length of the line segment AB?

3. (a) If a and b are real numbers such that $|5a + 2| + |b + 3| = 0$ what is the numerical value of ab?

   (b) If p, q and r are real numbers not all zero prove that $p + q + r$ and $pq + qr + rp$ cannot both be zero.

(See other side)

4. (a) The x-axis, the y-axis and the line whose equation is $4x - 3y + 10 = 0$ are all tangent to a circle whose centre is in the first quadrant. What are the coordinates of that centre?

(b) Prove that for every acute angle θ

$$\frac{4}{\sec^2\theta} + 9 \geq 12\cos\theta.$$

5. If m and n are integers prove that $\dfrac{m^2n^2 + m^2n + mn^2 + mn}{4}$ is an integer.

6. Find the area of the quadrilateral whose vertices are the four points at which the parabola with equation $y = 3x^2 - 2$ meets the circle whose equation is $x^2 + y^2 = 2$.

7. (a) Find all roots of $2x^3 - 3x^2 - 3x + 2 = 0$ given that 2 is a root of this equation.

(b) A polynomial has a remainder of 3 when divided by $x - 1$ and a remainder of 7 when divided by $x - 5$. What is the remainder when the polynomial is divided by $(x-1)(x-5)$?

8. From the set of integers from 1 to 3000 inclusive a subset of 2001 integers is chosen. Prove that this subset contains three integers whose product is divisible by 12.

9. A straight line with positive slope has exactly one point in common with the parabola whose equation is $y = x^2 + 2x + 2$ and also exactly one point in common with the parabola whose equation is $y = -x^2 + 2x - 2$. What is the equation of this line ?

10. In triangle ABC the lengths of all sides are integers, $CA = 6$ and $\angle BCA = 60^\circ$. If triangle ABC is not equilateral what are the lengths of AB and BC ?