2023 Manitoba Mathematical Competition Solutions

February 2023

1. (a) If $x = 4$, find the value of $\sqrt{x + \sqrt{6x+1}}$.

   (b) Find a pair of distinct positive integers x and y such that $\frac{1}{x} + \frac{1}{y} = \frac{1}{2}$.

Solution:

(a) $\sqrt{x + \sqrt{6x+1}} = \sqrt{4 + \sqrt{25}} = \sqrt{4+5} = 3$.

(b) 3 and 6.

Comments:

In general $\frac{1}{a(a+b)} + \frac{1}{b(a+b)} = \frac{1}{ab}$ so in particular $\frac{1}{n}$ is the sum of $\frac{1}{n+1}$ and $\frac{1}{n(n+1)}$, for any n. Thus (b) can be obtained analytically in this fashion taking $n = 2$. But of course it can also easily be solved directly by inspection.

2. (a) A set of consecutive integers has an average of 37. The third number and the sixth number, in sequence, have a sum of 19. Find the largest of the numbers.

Solution: Let x be the smallest number. We are given $(x + 2) + (x + 5) = 19$, so $2x + 7 = 19$. Therefore $x = 6$.
The average of consecutive integers is the one in the middle (if this is an integer). Since $37 = 6 + 31$, and 37 is the middle number the largest number must be $37 + 31 = 68$.

(b) Susan wrote two consecutive positive integers on a piece of paper. She then multiplied each of them by two and squared the results. The difference between the resulting squares was 84. What were the original numbers?

Solution: $(2x + 2)^2 - 2x^2 = 8x + 4 = 84$, so $x = 10$.
The original numbers were 10 and 11.

3. (a) Azeen drove his car for 160 km. For the first 40 km he travelled at an average speed of 50 km/h. For the rest of the trip he averaged 100 km/hr. What was his average speed for the entire trip?

(b) Glenda wrote three mathematics tests and had marks over 50 on all of them. She calculated her average mark and got a result of 48. She knew this could not be right. When she inspected her work she discovered that she had exchanged the two digits of her lowest mark. She then correctly determined her average to be 57. What was her lowest mark?

Solution:

(a) The first 40 km took $\frac{40}{50} = 0.8$ hours; the next 120 took $\frac{120}{100} = 1.2$ hours.
Azeen travelled 160 km in 2 hours so his average speed was $\frac{160}{2} = 80$ km/h.

(b) Let Glenda's lowest mark be $10a + b$ Using the transposed version, $10b + a$, she obtained a sum of $3 \cdot 48 = 144$ and using the correct version she obtained a sum of $3 \cdot 57 = 171$. The difference, of course, was

$$10a + b - (10b - a) = 9(a - b) = 171 - 144 = 27.$$

Therfore, $a - b = \frac{27}{9} = 3$. Since all marks were above 50 and the correct average was 57, the lowest mark is between 50 and 57. Therefore $a = 5$, and so $b = 2$.
Glenda's lowest mark was therefore 52.

4. (a) Find all ordered pairs (m, n) where m and n are integers satisfying

$$(m-2)^2(n+1) = 72.$$

(b) The digits 1, 2, 3, 4 can be arranged (without repetitions) to form 24 different four digit numbers. What is the sum of those numbers?

Solution:

(a) The possibilities may be tabulated as follows.

$(m-2)^2$	$m-2$	$n+1$	m	n
1	± 1	72	$3, 1$	71
4	± 2	18	$4, 0$	17
9	± 3	8	$5, -1$	7
36	± 6	2	$8, -4$	1

There are 8 resulting pairs:

$$(3, 71),\ (1, 71),\ (4, 17),\ (0, 17),\ (5, 7),\ (-1, 7),\ (8, 1),\ (-4, 1)$$

(b) Stack the 24 numbers and add down columns. Each column consists of 6 of each of the four possible digits and so adds to $6(1 + 2 + 3 + 4) = 60$. So the resulting sum of the described numbers is $60000 + 6000 + 600 + 60 = 66660$.

5. In each case, solve for x.

(a) $\dfrac{\sqrt{2}-1}{\sqrt{x}-1} = \dfrac{\sqrt{x}-1}{\sqrt{2}+1}$

(b) $\left(x^2-3x\right)^2 - 8 = 2\left(x^2-3x\right)$

Solution:

(a) $(\sqrt{x}-1)^2 = (\sqrt{2}-1)(\sqrt{2}+1) = 1.$
So $\sqrt{x}-1 = \pm 1$. Therefore $\sqrt{x} = 2$ or 0 and so $x = 4$ or 0.

(b) Let $y = x^2 - 3x$. Then

$$\begin{aligned} y^2 - 8 &= 2y \\ y^2 - 2y - 8 &= 0 \\ (y-4)(y+2) &= 0 \\ y &\in \{4, -2\} \end{aligned}$$

If $y = 4$ then $x^2 - 3x = 4$, so $x^3 - 3x - 4 = (x-4)(x+1) = 0$ so $x = 4$ or -1
If $y = -2$ then $x^2 - 3x = -2$, so $x^3 - 3x + 2 = (x-1)(x-2) = 0$ so $x = 1$ or 2.

6. If p and q are odd integers, prove that

$$x^2 + 2px + 2q = 0$$

has no rational roots.

Solution: Assume the equation has rational roots. Then the discriminant $4(p^2 - 2q)$ must be a perfect square. Since 4 is square, $p^2 - 2q$ is also a perfect square, so there exists integer n such that

$$\begin{aligned} p^2 - 2q &= n^2 \\ 2q &= p^2 - n^2 \\ &= (p+n)(p-n) \end{aligned}$$

$2q$ is even so $p+n$ and $p-n$ cannot both be odd. But they differ by $2n$ and so have the same parity. It follows that both are even.
So $2q$ is divisible by 4. Therefore q is even, which contradicts the hypothesis about q. Therefore there are no rational roots for this equation.

Alternate #1: Suppose a, b are roots of the equation and suppose one is rational. Then so is the other (by, say, long division, or by the remainder theorem). Since the polynomial is monic, by the Rational Roots Theorem, $a, b \in \mathbb{Z}$.
Thus $(x-a)(x-b) = x^2 + 2px + 2q$, so $a + b = 2p \equiv 2 \pmod 4$ and $ab = 2q \equiv 2 \pmod 4$. (Modular arithmetic is not needed, it is used here as an abbreviation.)
The solutions modulo 4 to $a + b \equiv 2 \pmod 4$ are $(a, b) = (0, 2), (2, 0), (1, 1), (3, 3)$, in which cases $ab \equiv 0, 0, 1, 1 \pmod 4$ respectively so $ab \not\equiv 2 \pmod 4$. Therefore there are no rational roots to the equation.

Alternate #2: Assume that there is a rational root. Let it be $x = \frac{a}{b}$ where a and b are relatively prime integers. Then $\frac{a^2}{b^2} + 2p\frac{a}{b} + 2q = 0$ which gives us

$$a^2 + 2pab + 2qb^2 = 0.$$

Since the last two terms are divisible by 2 it is easily seen that a^2 is also divisible by 2. If a^2 is even, then a is even. Let $a = 2k$. Then $4k^2 + 4pkb + 2qb^2 = 0$. Dividing by 2 we get: $2k^2 + 2pkb + qb^2 = 0$.
Since the first two terms are even it is easily seen that qb^2 is even. Since q is odd, b^2 must be even. Therefore b is even.
But we have reached a contradiction. Since a and b are relatively prime, they cannot both be even. So no rational root exists.
(This is based on Euclid's classical proof about the irrationality of the square root of 2.)

Alternate #3: Suppose there are rational roots a and b. Since the polynomial is monic, the Rational Root Theorem tells us that the roots are integers. The sum of the roots is $-2p$ and the product is $2q$. Since the product is even, at least one root is even. Since the sum is even, the other root is also even. Therefore the product is divisible by 4. But if 2q is divisible by 4, then q is even. Contradiction! Therefore the roots are irrational.

7. A corridor has seven lights numbered 1 to 7, each with its own switch. Initially, all lights are on. Adam makes a random selection of three lights which he turns off. Betty makes a random selection of three lights and changes their status (if a light was on she turns it off and vice versa). Chris then makes a random selection of three lights and changes their status. What is the probability that all the lights have been thus turned off?

Solution: Adam leaves 4 lights on and 3 lights off.

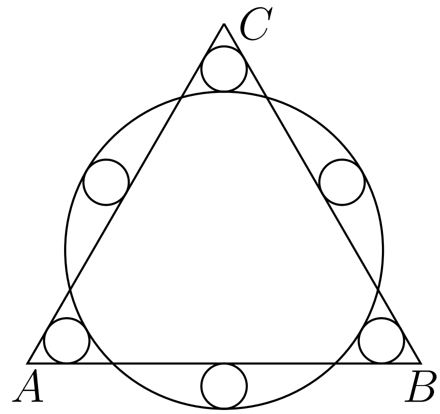
Assuming Chris leaves all lights off he must not have turned any lights on. Therefore when his turn came there were 3 lights on and 4 lights off, and he turned off the 3 that were on.

This leaves Betty in the middle, making a selection that changes 4 on and 3 off to 3 on and 4 off. To do this she must turn off 2 lights and turn on 1. She can select the lights to turn off in $\binom{4}{2} = 6$ ways and the one she wishes to turn on in 3 ways, so there are $6 \cdot 3 = 18$ ways to perform this task, whereas there are $\binom{7}{3} = 35$ ways to switch 3 lights, so the probability of making the required move, for her, is $\frac{18}{35}$.

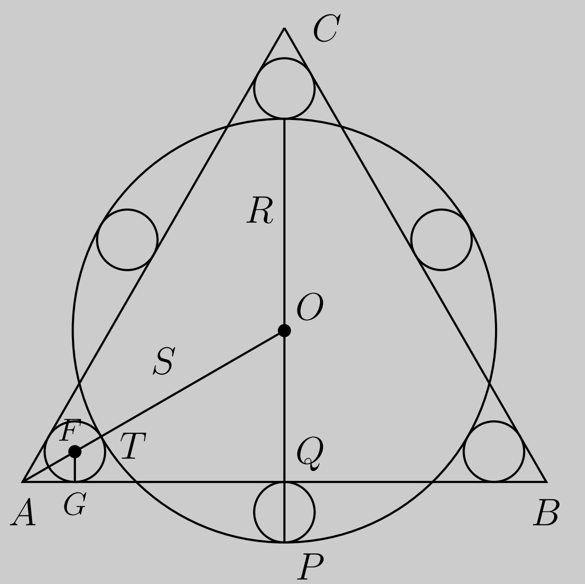
Chris must perfectly match the three remaining lights. The probability of this happening is $\frac{1}{35}$. These probabilities are independent of the choice of lights made by Adam.

So the probability that all lights are turned off at the end is $\frac{18}{35}\frac{1}{35} = \frac{18}{35^2}$ or $\frac{18}{1225}$.

8. Six small circles each have radius of one unit. Each is tangent to a larger circle and to one or two sides of an equilateral triangle $\triangle ABC$, arranged symmetrically, as shown. Find the radius of the larger circle.



Solution:



Let R be the radius of the big circle, O its centre, let $S = OA$, let P be the point on the big circle directly below O in the diagram. Let Q be the point of intersection of AB and OP and let T be the point of intersection of OA and the big circle.
Because $\triangle AOQ$ is $30 - 60 - 90$, $OQ = \frac{1}{2}S$.
Let F be the center of the small circle between A and O and let G be where that circle meets AB. Then

$$\begin{aligned} FG &= 1 \\ FT &= 1 \\ AF &= 2 \text{ (again by } 30 - 60 - 90 \text{ triangle)} \\ \therefore AT &= 3. \end{aligned}$$

So we have (considering OA) $S = R + 3$ and (considering OB) $R = \frac{S}{2} + 2$.
Therefore $R = \frac{R+3}{2} + 2$, so $2R = R + 3 + 4 = R + 7$.
So $R = 7$.

9. For how many positive integers $n \leq 100$ is the fraction $\dfrac{17n+4}{4n+3}$ reducible?

(Note: A fraction is reducible if it is is equivalent to another fraction with smaller denominator. For example, $\dfrac{4}{6}$ is reducible because it is equivalent to $\dfrac{2}{3}$ whereas $\dfrac{5}{6}$ is not reducible.)

Solution: If the fraction is irreducible then both $17n + 4$ and $4n + 3$ are divisible by some prime p. Then so is $17(4n + 3) - 4(17n + 4) = 35$. Therefore $p = 5$ or 7.
First consider the list of numbers n where $17n + 4$ and $4n + 3$ are both divisible by 7. Observe that n has this property. Further, if n has the property so must $n + 7$ as adding 7 to n increases $17n + 4$ by $17 \cdot 7$ and $4n + 3$ by $4 \cdot 7$. Adding $1, 2, 3, 4, 5$ or 6 gives differences not divisible by 7. The list is therefore $1, 8, 15, 22, \ldots, 99$.
Similarly for $n = 3$ both expressions are divisible by 5, and adding 5 (but not $1, 2, 3, 4$ to n preserves this property, so the list of numbers n where $17n + 4$ and $4n + 3$ are divisible by 5 is $3, 8, 13, 18, \ldots, 98$.
The first list has 15 numbers, the second has 20 and they overlap in the three numbers $8, 43, 78$. So $\dfrac{17n+4}{4n+3}$ is reducible for $15 + 20 - 3 = 32$ values of n, $1 \leq n \leq 100$.

Alternate #1: Starting from the reduction to considering prime factors 5 and 7 we require $4n + 3 \equiv 17n + 4 \equiv 0 \pmod{p}$.
Modulo 5 both expressions reduce to $n \equiv 3 \pmod{5}$ and the solution in $\mathbb{Z}$ is $n = 5h+3$, and within the given range: $3, 8, 13, 18, 23, 28, ...93, 98$, a total of 20 values.
Modulo 7 both expressions reduce to $n \equiv 1 \pmod{7}$ and the solution in $\mathbb{Z}$ is $n = 7k+1$, and within the given range: $1, 8, 15, 22, 29, ...92, 99$, a total of 15 values.
The two lists intersect in values $n = 8+35k$, or $8, 43, 78$. Therefore by inclusion/exclusion the expression is irreducible for $20 + 15 - 3 = 32$ values of n in this range.

Comments:

While this problem could be solved exhaustively by a very determined arithmetician, this would consume an inordinate amount of time. The challenge is to condense the work by avoiding longish arithmetic with growing numbers with systematic analysis.

10. The edges of a rectangular solid are all of integer lengths. Let p and q be odd prime numbers. Exactly five such solids have a volume of $2pq$; their dimensions are $2 \times p \times q$, $1 \times 2p \times q$, $1 \times p \times 2q$, $1 \times 2 \times pq$ and $1 \times 1 \times 2pq$.

(a) How many possible solids are there if the volume is to be $4pq$?

(b) How many possible solids are there if the volume is $2^{2n}pq$ where n is a positive integer? Express your answer in terms of n.

Solution: **Lemma**: The number of ways of distributing n identical objects into m distinct baskets is $\binom{n+m-1}{m-1}$. (A standard way to count such distributions is to code them as permutations of $m-1$ markers of one type—representing dividers between the baskets, in a line, and n markers of another type—representing the objects.)

Solution of the question (focussing on (b) as it suffices to cover (a)):
Case 1: One side of the solid is divisible by p and the other side by q.
Place a p into one basket, a q into another and a 1 into a third. Then distribute the $2n$ 2s into the three baskets. This can be done $\binom{2n+3-1}{3-1} = \binom{2n+2}{2} = 2n^2 + 3n + 1$ ways.
Case 2: One side of the solid is divisible by pq Place a pq into one basket, a 1 into each of 2 other baskets. Then distribute the $2n$ 2s into the three baskets. This can be done $\binom{2n+3-1}{3-1} = \binom{2n+2}{2} = 2n^2 + 3n + 1$, but this double-counts since the two baskets starting with 1 can be interchanged, except when the two baskets receive the same number of 2s (which will be when they both have $0, 1, 2, \ldots,$ or n, that is, $n + 1$ cases.
So this case accounts for $\frac{2n^2+3n+1+(n+1)}{2} = n^2 + 2n + 1$ ways.
By rule of sum there are $(2n^2 + 3n + 1) + (n^2 + 2n + 1) = 3n^2 + 5n + 2$ such solids.

Comments:

A challenging counting question in which management of cases presents a significant hurdle. It is unusual for us to place this type of enumeration task in the #10 position on the MMC but this one rose to that level, a decision vindicated by the evident difficulties students faced in their submitted solutions.

The lemma presented in our solution is a task-reducing device but not strictly necessary in the solution. Student attempts utilized different ways of keeping track of where the various 2s appear in a factorization.

A potential quibble over wording: the question did not specify that p, q are distinct, which of course makes a difference. Markers would have given full marks for a complete and correct solution under either intepretation, but the ambiguity in the question did not appear to impact any student's solution.

No students presented a flawless solution, and none received more than 4 marks on the question except for two who received 9 marks for complete, correct solutions with minor omissions or imprecision in explanation of logic. In one case the student cites another page for steps ... but the cited page was not submitted (perhaps the back of a page was not included in the submission? Reminder: both fronts and backs of pages are QR-coded for the purpose of properly loading into Crowdmark all work on a problem. Work cannot be credited if it is not submitted.)