2021 Manitoba Mathematical Competition
SOLUTIONS

General comments:

1. This draft of the solutions is tentative. If you see errors or alternative approaches that you think should be shown please let us know.

2. This year was exceptional because of the covid lockdown; the contest was distributed in a heterogeneous fashion to accommodate the various conditions of the schools. Questions were selected with the notion that a slightly different demographic may be writing, with some students "trying out" the experience of mathletics.

3. Our top student prizes are awarded in a different fashion by the MAMT, though our district prizes, funded by the Winnipeg Actuaries' Club, are awarded in the usual fashion this year. Teachers were additionally asked to judge and report to us on a student-by-student basis whether each student had written in a sufficiently rigorously invigilated fashion to be eligible for prizes. Thus not all students were in the prize pool.

4. Because of the demands of online teaching and the unusual formats in which several schools' papers arrived (requiring a great deal of pre-processing in order to use CrowdMark) we found ourselves without the regular cohort of markers and relied instead on a few volunteers, mainly graduate students, and the marking got started very late, with final rankings finished only a few weeks before the end of the school year at the end of June.

5. because we'd mixed things up in so many ways it was decided to release links to marked papers to the teachers so that it would be possible (if they deemed it helpful) for teachers to review their papers with marker comments, something quite unusual for higher-level math competitions. But the governing idea was that, because we had to treat the contest less formally this year, we would make it into more of an educational experience than usual. Markers were told this would be happening, and were encouraged to be informative in their comments.

1. (a) What is the probability that a randomly chosen number between 1 and 50 inclusive will have a 4 as a digit?

   (b) Let $X = |(-100) \cdot (-99) \cdots (99) \cdot (100)|$ (the absolute value of the product of all the integers from -100 to 100), and let $Y = (-100)^2 + (-99)^2 + \cdots + 99^2 + 100^2$. Which number is larger, X or Y? (mention a reason).

Solution:

(a) All 10 of the numbers in the forties will have a 4, as will each of 4, 14, 24, and 34. So the probabilty is $\frac{14}{50} = \frac{7}{25}$.

(b) X is a product, one of whose factors is 0. Therefore $X = 0$ whereas Y is clearly strictly positive. Therfore Y is larger.

Discussion:

(a) In (b) a minority (but a significant one) of students made the observation that 0 is a factor in X. The most common marks were around 5 out of 10.

2. Find the area enclosed by the graph of

   (a) $x^2 + y^2 = 9$

   (b) $|x| + |y| = 3$

Solution:

(a) Circle of radius 3. Area enclosed: $\pi \cdot 3^2 = 9\pi$.

(b) Area of four congruent triangles—one in each quadrant—$4\left(\frac{1}{2}bh\right) = 2(3 \cdot 3) = 18$.

Discussion:

- Alternatively (b) is a square of side $3\sqrt{2}$.
- The most common mark for this question was 0 about 15% received full marks.

3. (a) If y is 2 more than x, and

$$z = \frac{\frac{y}{x} + \frac{x}{y} + 2}{\frac{y}{x} - \frac{x}{y}}$$

then how much more than x is z?

(b) Solve for x:

$$\frac{x^3 + x - 2}{x - 1} = 5$$

Solution:

(a) Multiplying top and bottom by xy to clear fractions we obtain

$$z = \frac{y^2 + x^2 + 2xy}{y^2 - x^2} = \frac{(x+y)^2}{(x+y)(y-x)} = \frac{x+y}{y-x} = \frac{x + (x+2)}{(x+2) - x} = x + 1.$$

It follows that z is 1 more than x.

(b) By polynomial long division (or synthetic substitution), $x^3 + x - 2 = (x-1)(x^2 + x + 2)$. Therefore we have $x \neq 1$ and $x^2 + x + 2 = 5$, whence $x^2 + x - 3 = 0$. The quadratic formula yields

$$x = \frac{-1 \pm \sqrt{13}}{2}$$

Discussion:

(a) In (a) it may be noted that neither x nor y can be 0 but this has no direct bearing on the answer. An ambitious student might also note that x cannot be equal to y but this is irrelevant since the first condition rules this out.

(b) In (b), however, it is important that $x = 1$ be eliminated.

(c) This question was relatively well done, with over half of participants receiving 5 or better.

4. (a) Apollo runs in a race with 10 runners where 5 distinct trophies (1st place, 2nd place etc.) are given to the top 5 winners, in how many different ways can the prizes be given if there are no ties, and Apollo must be one of the top 3 winners?

   (b) If n is a positive integer and $2n+1$ is a perfect square, show that $n+1$ is the sum of two consecutive perfect squares.

Solution:

(a) 3 ways to place Apollo and 9(8)7(6) ways to place 4 other people in the remaining winning positions. Hence, $3 \cdot 9 \cdot 8 \cdot 7 \cdot 6 = 9072$ ways.

(b) Write $2n+1 = s^2$. Since $2n+1$ is odd, then s must also be odd. Let $s = 2t+1$. Then

$$2n+1 = (2t+1)^2 = 4t^2 + 4t + 1$$

Cancelling, we have that $n = 2t^2 + 2t$ so

$$n+1 = 2t^2 + 2t + 1 = t^2 + (t+1)^2$$

as required.

Discussion:

(a) Students appeared to be stymied by both parts of this question. The majority of participants received 0 marks for it.

(b) It is a bit unusual for us to group half-questions of completely different natures and subject matter, but our intention was to provide as much variety as possible this year.

(c) Also it is unusual to include a proof in the half-question portion of the MCC – ordinarily questions 1 to 3 and sometimes 4 or 5 are marked mostly answer-only which would preclude such questions, but we kept (b) because we felt it was a simple enough argument to unlock and placed it early enough that even weaker students would be encouraged to explore it. Perhaps we misjudged students' preparation for this kind of exploration. A successful attack from scratch that does not follow the analytic proof shown might explore the first few odd squares and seek a pattern. Indeed, $3^2 = 2 \cdot 4 + 1$ and $4 + 1 = 1^2 + 2^2$. Then $5^2 = 2 \cdot 12 + 1$ and $12 + 1 = 13 = 2^2 + 3^2$. Then $7^2 = 2 \cdot 24 + 1$ and $24 + 1 = 25 = 3^2 + 4^2$. Further, $9^2 = 2 \cdot 40 + 1$ and $40 + 1 = 41 = 4^2 + 5^2$. One might guess (then verify) from this a relationship between s and t that is a little easier to infer than between n and t.

5. Solve the equation

$$\sqrt{x} + 3\sqrt[6]{x} = 4\sqrt[3]{x}$$

Solution: Let $x = t^6$ (for the first term to be defined we must have $x \geq 0$). Then

$$\begin{aligned} t^3 + 3t &= 4t^2 \\ t(t^2 - 4t + 3) &= 0 \\ t(t-1)(t-3) &= 0 \end{aligned}$$

The solution is $x \in \{0^6, 1^6, 3^6\} = \{0, 1, 729\}$.

Discussion:

(a) Students appeared to have real difficulty with this, and most did not seem familiar with the tactic of making a convenient simplifying substitution. Perhaps radicals of degrees greater than 2 are a little unnerving for many. Somewhat less than 10% of students received full or near-full marks.

6. Consider the set of numbers x satisfying $0 \leq x \leq 1$. Two numbers are randomly chosen from this interval. What is the probability that they differ by less than 0.2?

Solution: Let x and y be the two numbers and consider the point (x, y) in the square $0 \leq x, y \leq 1$ in the plane, which has area 1.
The probability that $|x - y| \leq 0.2$ is the area of the region consiting of points (x, y) within this square satisfying this inequality, i.e., between the lines $y = x + 0.2$ and $y = x - 0.2$. The remaining region consists of two triangles of base and height 0.8, so the region not falling between these lines has area $2 \cdot \frac{1}{2} \cdot 0.8^2 = 0.64$.
It follows that the required probability is $1 - 0.64 = 0.36$.

Discussion:

(a) A small number of students successfully solved (or nearly solved) this problem by an algebraic analysis from scratch without using a geometric metaphor. Others attempted the geometric solution but committed minor errors in an otherwise correct approach, leading to partial loss of marks. The vast majority received close to 0, with some partial credit assigned to those who'd made some partial calculations which in themselves were correct but did not make significant progress toward a complete answer.

7. A wheel with radius 2 meters rolls around the outside of a regular convex heptagon (that is, a 7-sided polygon with equal sides and angles) of side length 3 meters. Determine the length of the path traced out by the centre of the wheel.

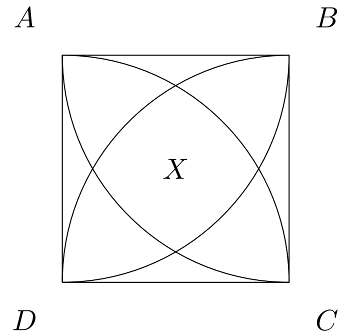
Solution: As the wheel rolls across the side of the heptagon the centre of the wheel moves parallel, so this adds 3m per side to the path. At a corner, the wheel transitions from one side to the next, all while maintaining contact with the corner between those sides. The centre of the wheel is always 2 m from the side. This causes the centre of the wheel to trace out an arc from a circle of radius 2 m of an angle which is supplementary to the interior angle of the heptagon, which, since it is regular and convex, is $\frac{2\pi}{7}$. Together all seven of these arcs trace out a circle, so the path travelled has length, $7 \cdot 3 + 2\pi \cdot 2 = 21 + 4\pi$ m.

Discussion:

(a) Student success on this problem was slightly better than on question 6.

(b) Of those completing this problem there was a tendency to neglect to state units, which ought to be essential but for the sake of separating those performing well on the more advanced problem solving, this was noted but not penalized; it may have been used as a consideration in breaking ties.

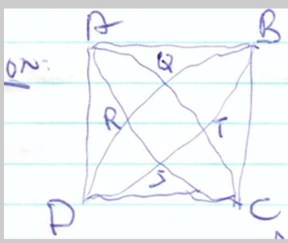
8. $\square ABCD$ is a square with sides of length 2. At each vertex a quarter circle of radius 2 is drawn as shown. Find the area of the intersection, X, of the four semicircles.

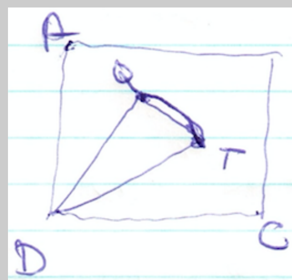


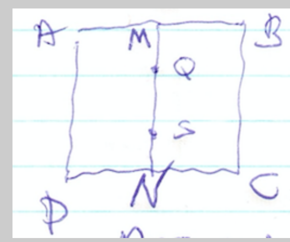
Solution: Let the quarter circles intersect at points Q, R, S and T.Since $\triangle QCD$ is equilateral $\angle QDC = 60°$. Similarly $\angle TAD = 60°$. It follows that

$$\angle QDT = 60° + 60° - 90° = 30°.$$

The area of sector QDT is thus $\frac{30}{360}\pi(2^2) = \frac{\pi}{3}$.







At the same time the area of $\triangle QDT$ is $\frac{1}{2} \cdot 2 \cdot 2 \sin 30° = 1$, whence it follows that the area of the segment formed by the arc and chord QT is $\frac{\pi}{3} - 1$.
The required region consists of quadrilateral (square—by symmetry) $QRST$ and four such segments. Since $QS \perp RT$ the area of the quadrilateral is $\frac{1}{2}(QS)(RT)$.
Extending QS to meet AB at M and CD at N recall that $\triangle QDC$ and $\triangle ABS$ are equilateral of side 2, so $SM = QN = \sqrt{3}$, and $QS = RT = 2\sqrt{3} - 2$.
Therefore the required area is

$$\frac{1}{2}(QS)(RT) + 4(\frac{\pi}{3} - 1) = \frac{1}{2}(2\sqrt{3} - 2)^2 + \frac{4}{3}\pi - 4 = 4 - 4\sqrt{3} + \frac{4}{3}\pi.$$

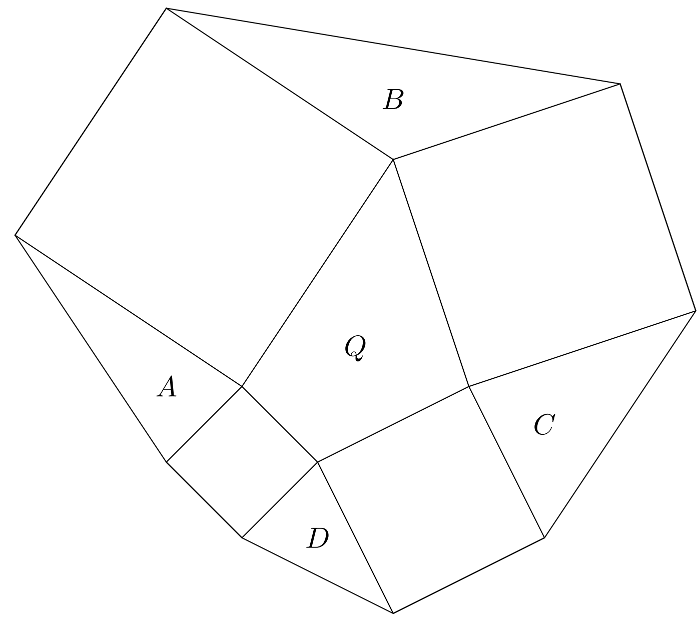
Discussion:

(a) Several students made progress but either committed serious errors or got lost partway and received only partial credit. The most common mark was 0; only one student received above 5 marks – losing only one perfunctory mark for committing a minor presentation sin.

(b) In this draft we have not yet replaced the poser's rough sketch with a proper graphic, but it should be clear enough.

9. On each side of a convex* quadrilateral with area Q a square is constructed. Segments are added between near vertices of neighbouring squares, creating a figure consisting of Q, surrounded by four squares, alternating in cyclic order with four triangles of areas A, B, C and D (see diagram). Prove that $A+C = B+D = Q$.

   ***convex** means that any line segment whose endpoints are contained in Q is entirely contained in Q.



Solution: Cut Q into two triangles by a segment from the vertex x on B to the vertex y on D. Rotating A by 90° clockwise around its common vertex z with Q gives a triangle A' with one side coinciding with xz and having common third vertex y' which is the reflection of y in xz. $\triangle xyz$. Therefore $A = |\triangle xyz|$. Similarly C is equal to the other triangle making up Q. It follows that $A+C = Q$. The same argument shows that $B + D = Q$.

Discussion:

(a) Similarly to number 8, only one student received above 5 marks – and received full credit. Many students appeared to correctly perceive that Q can be split in two ways into triangles whose areas match A, B, C, D but most were unable to correctly demonstrate the required equalities.

10. Find all ordered triples of polynomials $p(x), q(x), r(x)$ so that the following conditions hold:

- $p(x)q(x)r(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^3$
- $p(1) = q(1) = r(1)$
- $\deg(p) \le \deg(q) \le \deg(r)$ (deg(f) denotes the degree of polynomial $f(x)$)

Solution: Solution Let $F(x) = (1 + x + x^2 + x^3 + x^4 + x^5)^3$.
Since $F(x) = p(x)q(x)r(x)$ then $6^3 = F(1) = p(1)q(1)q(1) = p(1)^3$.
So then $p(1) = q(1) = r(1) = 6$. Notice that $1+x+x^2+x^3+x^4+x^5 = (1+x)(1+x+x^2)(1-x+x^2)$ is factored into its irreducible factors. To make our polynomials we will need to distribute the 3 copies of each of $1+x$, $1+x+x^2$, and $1-x+x^2$. Respectively, each of these polynomial factors when evaluated at 1 gives 2, 3, and 1. In order to have $p(1) = q(1) = r(1) = 6$ the factors $1 + x$ and $1 + x + x^2$ must be equally distributed, one to each polynomial.
There remain the 3 factors $1 - x + x^2$ which won't change the value of $p(1)$, $q(1)$, or $r(1)$ as we distribute them. There are three options: (1) each polynomial gets one factor of $1-x+x^2$, (2) $q(x)$ gets one factor of $1 - x + x^2$, and $r(x)$ gets two factors of $1 - x + x^2$, and (3) $r(x)$ gets all three factors of $1 - x + x^2$. This will ensure the degree requirement is satisfied.
That is the three solutions are:
Solution 1:

$$p(x) = (1 + x)(1 + x + x^2)(1 - x + x^2)$$
$$q(x) = (1 + x)(1 + x + x^2)(1 - x + x^2)$$
$$r(x) = (1 + x)(1 + x + x^2)(1 - x + x^2)$$

Solution 2:

$$p(x) = (1 + x)(1 + x + x^2)$$
$$p(x) = (1 + x)(1 + x + x^2)(1 - x + x^2)$$
$$p(x) = (1 + x)(1 + x + x^2)(1 - x + x^2)^2$$

Solution 3:

$$p(x) = (1 + x)(1 + x + x^2)$$
$$p(x) = (1 + x)(1 + x + x^2)$$
$$p(x) = (1 + x)(1 + x + x^2)(1 - x + x^2)^3$$

Discussion:

(a) The performance on Question 10 was very similar to the previous two questions. Two students received more than 5 marks, one getting 10/10 and the other receiving 8. Very few got more than 0. One student attempted to solve over $\mathbb{C}$ and showed good familiarity with roots of unity but had trouble enumerating cases and meeting condition (2).

(b) Our last question is generally intended to be something few students can solve, but perhaps one or two will complete or nearly complete correctly. This one appears to have been judged correctly.