2019 Manitoba Mathematical Competition Solutions

1. (a) Find an integer n satisfying

$$\frac{3}{7} < \frac{n}{6} < \frac{4}{7}$$

(b) Will, Bob, Sara and Ida wrote integers on the blackboard.

- Bob's number was 15 less than Will's number.
- Sara's number was equal to the square of Bob's number.
- Ida's number was one-half of Sarah's number.

If Ida wrote the number 72, what possible numbers might Will have written?

Solution:

(a) Multiply by 42 to get

$$18 < 7n < 24.$$

The only multiple of 7 between 18 and 24 is 21. So $n = 3$.

(b) No. If Will wrote x, then

$$\begin{aligned} \frac{(x-15)^2}{2} &= 72 \\ (x-15)^2 &= 144 \\ x - 15 &= \pm 12 \\ x &= 3 \text{ or } 27 \end{aligned}$$

Comments:

- Many students failed to include the negative square root in (b).

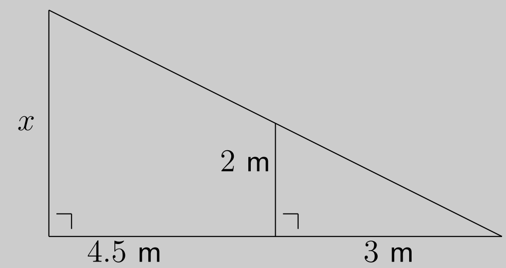
2. (a) The head of a fish is 10 inches long; the tail is as long as the head plus one-half of the body; the body is as long as the head and tail together. How long is the fish?

   (b) A man who is 2 m tall stands 4.5 m from a street light. His shadow is 3 m long. How high is the street light?

Solution:

(a) Let b and t be the lengths of the body and tail in inches respectively. We have $t = 10 + \frac{b}{2}$ and $b = 10 + t$, so $b = 10 + 10 + \frac{b}{2}$ or $b = 40$, so $t = 40 - 10 = 30$. The length of the fish is thus $10 + 30 + 40 = 80$ inches (big fish!).

(b) By similar triangles: $\frac{x}{7.5} = \frac{2}{3}$ so $3x = 15$, and $x = 5$ so the street light is 5 m high.



Comments:

- Some students noticed a different set of similar triangles, giving $2 : 3 = (x - 2) : 4.5$. Some might consider this conceptually cleaner than the one given here, though both require about the same amount of work.

3. (a) How many ordered triples of positive integers (a, b, c) can be formed so that $abc = 12$? For example, one such triple is $(a, b, c) = (2, 2, 3)$.

   (b) Prove that there is no ten-digit prime number which uses every decimal digit exactly once.

Solution:

(a) Taking $0 < a \leq b \leq c$ we quickly obtain, through ad hoc analysis, the triples $(1, 1, 12)$, $(1, 2, 6), (1, 3, 4), (2, 2, 3)$. There are 3 permutations of the first and last triples, and 6 permutations of the other two, for a total of 18.

(b) The sum of the digits of such a number is $45 = 5 \cdot 9$, and so by a well-known test it will be divisible by 9 and so also by 3. The only prime divisible by 3 is 3 itself, which is not a ten-digit number, so such a number cannot be prime.

Comments:

- A few students seemed not to understand how the word "ordered" affects the counting in (a).
- When making an argument, mention all salient points of logic. It was common for students to calculate the sum of the digits but not to mention the critical point: that 45 is divisible by 9 (or 3). For many things showing a calculation amounts to explaining one's logic but in this case one should mention what it is about the calculation that permits the conclusion.

4. Solve each of the following equations

(a) $\dfrac{1}{x+1} + \dfrac{1}{x+3} = 1$

(b) $(2x + y - 3)^2 + (x - 2y + 3)^2 = 0$

Solution:

(a) Clearing fractions gives

$$\begin{aligned} x + 3 + x + 1 &= (x + 1)(x + 3) \\ 2x + 4 &= x^2 + 4x + 3 \\ x^2 + 2x - 1 &= 0 \end{aligned}$$

The quadratic formula (or completing the square) gives $x = -1 \pm \sqrt{2}$.

(b) Observe that squares are never negative, so each square must be 0 in order for this equation to be valid. So we have

$$\begin{aligned} 2x + y - 3 &= 0 \\ x - 2y + 3 &= 0 \end{aligned}$$

—a system of two linear equations in two variables, which is easy to solve (add twice the first equation to the second to eliminate y, for example), and the resulting solution is $(x, y) = \left(\frac{3}{5}, \frac{9}{5}\right)$.

Comments:

- In both parts there is potential to get stuck in a morass of calculations — especially (b) if the student begins by expanding, which is very likely.
- A common serious error in (b): writing $(2x + y - 3)^2 = -(x - 2y + 3)^2$, then calculating the square root of both sides and proceeded from there. Most arrived at the correct answer, but the method is spurious, and it was largely luck that partial conclusions, incorrectly derived, could be massaged into a correct solution.

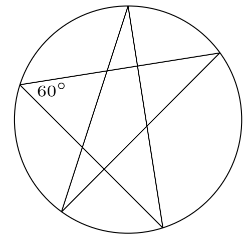
  The first problem is that one side is positive and the other negative. One ought to conclude at this point that both are 0 but these students seemed to miss this.

  The two main approaches from here:

  - Some took the positive root of both sides, ignoring the problem of signs (or a variant in which the negative root is obtained on the right side). This is simply incorrect. This gives a linear equation, which yields a relation between x and y. Using this "successful" solutions then eliminated one of x and y in the original equation and solved from there. Tedious, but a correct answer is obtained.
  - Others wrote down four cases, taking into account both possible signs of the root on both sides. Again, this is an error unless one explicitly invokes

that both sides are zero—and then ... why bother with the cases? Each case, of course, yields only a linear relation between x and y. "successful" solutions then combined results from two cases to obtain, and solve, linear systems. This is completely fallacious. When a solution splits into cases they are mutually exclusive; their separate results can't then be combined!

5. (a) Which number is greater, $A = \sqrt{6} + \sqrt{10}$ or $B = \sqrt{5} + \sqrt{12}$?



(b) Imogene draws an irregular five pointed star inside a circle. She measures the angles at the five points and finds that no two are equal. The largest one measures $60°$. Show that the second largest is more than $30°$.

Solution:

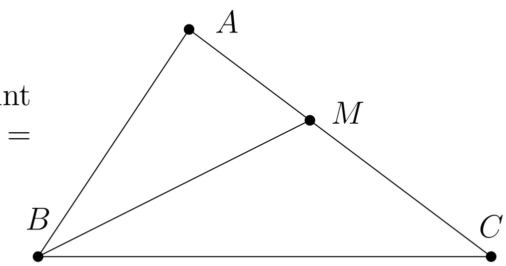
(a) $A^2 = (\sqrt{6} + \sqrt{10})^2 = 16 + 2\sqrt{60}$ and $B^2 = (\sqrt{5} + \sqrt{12})^2 = 17 + 2\sqrt{60}$. Hence $B^2 > A^2$. Since $A, B > 0$ we therefore have $B > A$.

(b) Each point measures half of the degree measure of the arc it intercepts. The 5 intersepted arcs form a circle of $360°$, so the 5 points have a total of $180°$. After the largest the remaining four have a total of $180 - 60 = 120$. If they were all less than $30°$ this would be impossible. Therefore the second largerst arc is $\geq 30°$. Since the arc lengths are all different it must be $> 30°$ (otherwise all four are equal to $30°$.)

Comments:

- In (a) some students were finding one-decimal approximations to roots by trial and error. This works but is a tedious way to tackle problems like this; not a good strategy because you cannot know beforehand what amount of precision is needed.
- In (b) many students found the sum of angles formula for the points of a pentagram but often used spurious reasoning such as using special cases (and failing to justify the generalization).

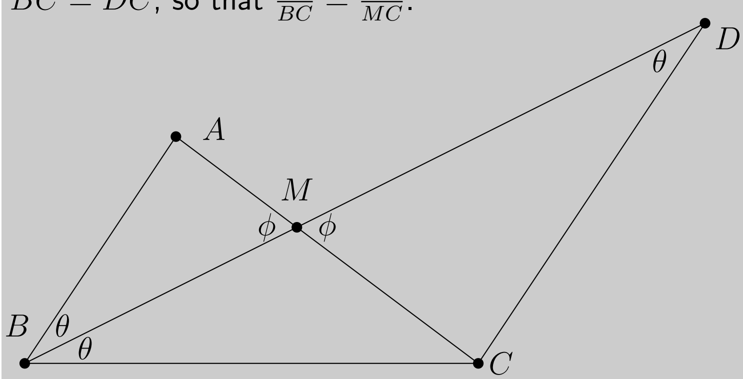
6. In triangle ABC, $\angle ABC$ is acute and M is a point on AC. If $\angle ABM = \angle CBM$ prove that $\frac{BA}{BC} = \frac{MA}{MC}$.



Solution: Construct a line through C parallel to AB, which meets BM at D. Then $\angle ABM = \angle CDM$ ($= \theta$) and $\angle AMB = \angle DMC$ ($= \phi$).

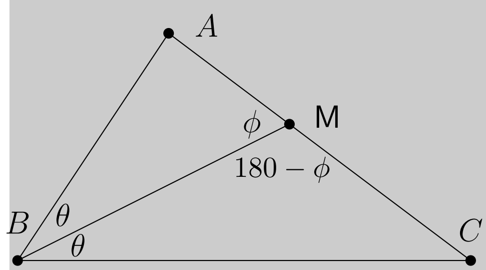
Therefore $\triangle ABM \sim \triangle CDM$.

Thus $\frac{BA}{DC} = \frac{MA}{MC}$. Now $\angle ABM = \angle CBM$ ($= \theta$) implies that $\triangle BCD$ is isosceles with $BC = DC$, so that $\frac{BA}{BC} = \frac{MA}{MC}$.

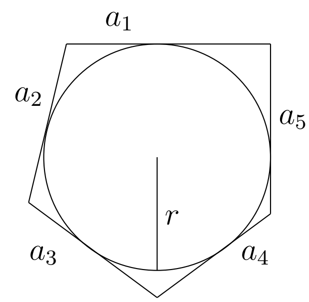


Alternate #1: By the law of sines, $AB = \frac{\sin\phi}{\sin\theta}AM$ (see diagram) and $BC = \frac{\sin(180-\phi)}{\sin\theta}CM = \frac{\sin\phi}{\sin\theta}CM$. Therefore,

$$\frac{AB}{BC} = \frac{\frac{\sin\phi}{\sin\theta}AM}{\frac{\sin\phi}{\sin\theta}CM} = \frac{MA}{MC}.$$



7. A convex polygon with n sides has an inscribed circle of radius r, and side-lengths $a_1, a_2, \ldots, a_n$, in cyclic order.



   (a) Suppose $n = 5$, and the sides of lengths a_1 and a_5 are perpendicular (shown). Find an expression for r in terms of a_1, a_2, a_3, a_4 and a_5.

   (b) Suppose n is even. Prove that the alternating sum

$$a_1 - a_2 + a_3 - \cdots - a_n$$

   is zero.

Solution:

(a) We build on the insight explained in (b) below, noting that the two segments at the given right angle are both equal to r. Therefore $a_1 - a_2 + a_3 - a_4 + a_5 = 2r$. Therefore $r = \frac{a_1 - a_2 + a_3 - a_4 + a_5}{2}$.

(b) Each side of the polygon is cut into two parts by the point of tangency with the circle. Observe that the two segments meeting at each vertex must be equal by symmetry. These cancel in the alternating sum of the sides to give zero.

Comments:

- The right insight knocks both parts apart relatively easy partial progress. It takes a bit of insight to see that (a) can be purchased by a modest investment of the method that solves (b).
- Another nice question using the same setup: suppose the perimeter of the polygon is P. What is its area in terms of P and r?

8. If $\sqrt{x-13}$ and $\sqrt{x+20}$ are both integers, find all possible values of x and prove that no other values are possible.

Solution: Let $N = \sqrt{x-13}$ and $N + k = \sqrt{x+20}$. We have

$$\begin{aligned}(N+k)^2 - N^2 &= 33\\ 2Nk + k^2 &= 33.\end{aligned}$$

Therefore k is a divisor of 33, so $k = 1, 3, 11$ or 33. (By construction $k > 0$). We proceed by cases:

$k = 1$: Then $2N + 1 = 33$ so $N = 16$. Putting $\sqrt{x-13} = 16$, $x - 13 = 256$, $x = 269$.

$k = 3$: Then $2N \cdot 3 + 3^9 = 33$ so $2N = 8$, or $N = 4$. $\sqrt{x-13} = 4$ gives $x = 29$.

$k = 11$: Then $2N + 1 = 33$ so $N = 16$. $22N + 11^2 = 33$ gives $2N + 11 = 3$—impossible (as $N > 0$).

$k = 33$: Then $66N + 33^2 = 33$ so $2N + 33 = 11$—again, impossible.

Therefore the only possible values are $x = 29$ and $x = 269$.

Comments:

- A good number of students gave correct, or partially correct, solutions using the fact that the list of the difference of consecutive squares gives the positive odd numbers in increasing order. Some did so without explicit mention, and others failed to adequately check all possibilities, losing some marks.

9. A child has 9 wooden blocks. Three of them have the letter A painted on each side. Three have the letter B and the other three have the letter C. She is asked to use 5 of these blocks to make a 5 letter sequence with the restriction that no two As can be consecutive. For example $CBBAC$ is an acceptable sequence but $ABAAC$ is not.

   How many acceptable sequences are there?

Solution:

Case 1: No As are used, which forces a permutation of either $BBBCC$ or $BBCCC$, of which there are $2\binom{5}{2} = 20$.

Case 2: Exactly one A is used, forcing a permutation of one of $ABBBC$, $ABCCC$ or $ABBCC$; using multinomial coefficients gives $2\frac{5!}{3!1!1!} + \frac{5!}{2!2!1!} = 2 \cdot 20 + 30 = 70$.

Case 3: Exactly two As are used. They can appear together in 4 ways: $AA***$, $*AA**$, $**AA*$ or $***AA$, so there are $\binom{5}{2} - 4 = 6$ acceptable ways to place the As; the remaining blocks are an unrestricted arrangement of Bs and Cs, so there are $6 \cdot 2^3 = 48$ such sequences.

Case 4: All three As are used, and the only arrangement of As is $A*A*A$, and Bs, Cs can be placed in the remaining squares in 2^2 ways.

It follows that the number of acceptable sequences is

$$20 + 70 + 48 + 4 = 142.$$

Comments:

- The main difficulties were distinguishing between permutations and combinations, and organizing work so that there was a complete, nonoverlapping set of cases.

10. Consider the equation

$$ax^2 + y^2 + 4xy + 2x + 2y = 0$$

where a, x, y are real numbers. For every real value of x there exists at least one real value of y. What are the possible values of a?

Solution: $y^2 + (4x + 2)y + (ax^2 + 2x) = 0$ is a quadratic in y with solution

$$y = \frac{-(4x+2) \pm \sqrt{(4x+2)^2 - 4 \cdot 1 \cdot (ax^2 + 2x)}}{2}.$$

For every real value of x we require that

$$\begin{aligned}(4x+2)^2 - 4(ax^2 + 2x) &\geq 0 \\ 16x^2 + 16x + 4 - 4ax^2 - 8x &\geq 0 \\ 4x^2 + 2x - ax^2 + 1 &\geq 0 \\ x^2(4-a) + 2x + 1 \geq 0\end{aligned}$$

For this to be nonnegative for all x, the following must be true:

(a) $4 - a > 0$, or $a < 4$;

(b) $2^2 - 4(4 - a) \leq 0$, or $a \leq 3$.

Both conditions hold whenever $a \leq 3$
Therefore for every value of x there is a real value of y if and only if $a \leq 3$.

Comments:

- Students found this one very challenging, apparently. Only two papers scored above 5 (out of a possible 10) on the question, and only one of these achieved a score of 10.