2017 Manitoba Mathematical Competition Solutions

1. (a) In how many ways can \$20 be changed into dimes and quarters, if at least one of each type of coin must be used?

Solution: Start with 80 quarters. We exchange some, but not all, of these for dimes. (By replacing 2 quarters at a time with 5 dimes.) Our choices are to exchange $2, 4, 6, 8, \ldots, 78$ quarters for dimes. That is 1, 2, 3, ..., 39 groups of 2. Thus there are 39 ways to perform the required task.

(b) A box contains 5 nickels and 5 dimes. Two coins are selected at random. What is the probability that the total value of the selected coins is less than or equal to 15 cents?

Solution: The total value is one of $\$.10$, $\$.15$ or $\$.20$. The probability of value $\$.20$, or selecting two dimes, is $\frac{5}{10} \cdot \frac{4}{9} = \frac{2}{9}$. Therefore the probability that the value is $\leq \$.15$ or, equivalently, $< \$.20$, is $1 - \frac{2}{9} = \frac{7}{9}$.

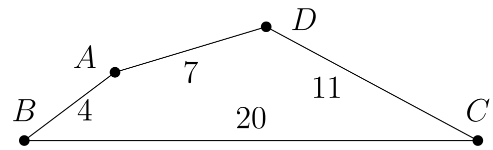
2. (a) On a standard 8×8 chessboard one of the 64 unit squares is chosen at random. Determine the probability that the chosen square has no edge in common with the perimeter of the chessboard.

Solution: The 8×8 board has a perimeter of 28 squares sharing an edge with the outside of the grid. This leaves 36 squares in the interior. Since each of the squares is equally likely to be chosen we get a probability of $\frac{36}{64} = \frac{9}{16}$.

(b) Determine the number of different two-letter words that can be formed using the letters of the words TRAP CARDS. (A two-letter word here consists of any pair of letters in a given order. A letter may be used twice only if it appears twice in the given phrase.)

Solution: To make up a two letter 'word' we just need to pick a letter and then second letter. Now, the two As are indistinguishable as are the two Rs. We have two possibilities, our 'word' uses the same letter twice or two different letters. If the same letter is used twice we have two possibilities, AA or RR. Otherwise we have 7 choices (T, R, A, P, C, D, S) for the first letter, 6 for the second letter. So we have $2 + 7 \times 6 = 2 + 42 = 44$ 'words'.

3. (a) The sides of quadrilateral $ABCD$ are:
$AB = 4$, $AD = 7$, $BC = 20$ and $DC = 11$.
Given that AC is an integer, find its value.



Solution: By the triangle inequality, $AC < 7 + 11$, $AC + 4 > 20$, so $16 < AC < 18$. It follows that $AC = 17$.

(b) Positive integers a, b, c and d are added three at a time. The sums are 93, 96, 105, and 114. What is the value of the largest of a, b, c and d?

Solution:

$$
\begin{aligned}
a + b + c &= 93 \\
a + b + d &= 96 \\
a + c + d &= 105 \\
b + c + d &= 114
\end{aligned}
$$

Adding,

$$3(a + b + c + d) = 408$$

So $a + b + c + d = 136$. Subtracting $a + b + c$ gives $d = 136 - 93 = 43$, etc, and the others are clearly smaller by the corresponding calculations. So the largest of the numbers is $d = 43$.

4. (a) Suppose a rectangular sheet of paper, when folded in half and cut along the fold, forms two smaller sheets of paper with the same ratio of length to width as the original sheet. Find the ratio of length to width. Justify your answer.

Solution: Let the original sheet have width x and length $2y$. Then $\frac{x}{2y} = \frac{y}{x}$. Hence $x^2 = 2y^2$, so $x = y\sqrt{2}$. The ratio of length to width is $\sqrt{2} : 1$.

(b) If the original sheet of paper, described above, has an area of $1\ \text{m}^2$ we call it an A0 sheet of paper. It can be folded in half and cut to form two A1 sheets. These can be folded in half and cut to form A2 sheets. Continuing this process we generate A3 and A4 sheets of paper. The A4 sheet of paper is the standard office paper used in most countries of the world. The A4 sheet of paper has dimensions 2^a metres by 2^b metres. Find a and b.

Solution: A1, A2, A3, and A4 sheets have areas of $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}$, and $\frac{1}{16}$ respectively. Let the A4 sheet have width x and length $x\sqrt{2}$. Then:

$$x(x\sqrt{2}) = \frac{1}{16}$$
$$x^2 = \frac{1}{16\sqrt{2}} = 2^{-\frac{9}{2}}$$

Therefore $x = 2^{-\frac{9}{4}}$ and $x\sqrt{2} = 2^{-\frac{7}{4}}$.
Thus $a = -\frac{9}{4}$ and $b = -\frac{7}{4}$.

5. (a) Both digits of a two digit number K (written in standard notation) are nonzero. When their order is reversed the new number is 45 less than K. Find all possible values for K.

Solution: Let $N = 10a + b$ be ab, where $1 \leq a, b \leq 9$. Then $(10a + b) - (10b + a) = 10(a - b) + (b - a) = 9(a - b) = 45$, so $a - b = 5$. Clearly this is equivalent to the original condition, so the possibilities are $(a, b) = (6, 1), (7, 2), (8, 3), (9, 4)$ and so N is one of $61, 72, 83$ or 94.

(b) N is an integer strictly between 10^3 and 10^4. Reversing the digits of N gives a number M also strictly between 10^3 and 10^4. Further, both M and N are divisible by 45. Find all possible values for N.

Solution: Since $10^3 < M, N < 10^4$ the last digits of M and N cannot be 0. Since both numbers are divisible by 45—a multiple of 5, their last digits can only be 0 or 5. Therefore the last digit—and so also the first digit—of both numbers is 5.
Let $N = 5AB5$ (written in standard notation). Then by the standard test for divisibility by 9, $5 + A + B + 5 = A + B + 10$ is divisible by 9. It follows that $A + B = 8$ or 17 and, again, this condition is equivalent to the original condition (since divisibility by 9 is equivalent to divisibility by 45 for numbers divisible by 5). It is now easy to enumerate all possibilities for (A, B), of which there are 11:

$$(A, B) \in \{(0, 8), (1, 7), (2, 6), (3, 5), (4, 4), (5, 3), (6, 2), (7, 1), (8, 0), (8, 9), (9, 8)\}.$$

So $N \in \{5085, 5175, 5265, 5355, 5445, 5535, 5625, 5715, 5805, 5895, 5985\}$.

Alternate #1: (b) sketch after $n = 5AB5$: We are looking for odd numbers between 5000 and 6000 which are divisible by 45. 5000/45 is a little bigger than 22, so the first number we seek is 45 x 23 = 5085. The numbers will form an arithmetic sequence with common difference 90, so we get 5085, 5175, . . . , 5985.

6. How many times must one flip a fair coin to guarantee that the probability of getting one or more runs of three consecutive heads or tails is 50% or more? Justify your answer.

Solution: We build a binary tree from a root representing a null string. Each branch is labelled either H or T (paths from the root represent sequences of outcomes of coin flips).
After three flips, we have built two strings with runs of three: 000 and 111. Mark each of these branches "1/8" and stop building them. You know that any parts of the tree built upon these will have a run of three, and the totality of each of those subtrees, at any subsequent level, will contain 1/8 of the "leaves" of the tree. Then build the other branches. At the next level you see 0111 and 1000. Mark each of these branches "1/16" and stop there – again you know they will each represent 1/16 of the subsequent tree. Thus far you have accounted for $1/8+1/8+1/16+1/16 = 3/8$ of the outcomes ... not quite 1/2.
Build the remaining branches out one more level and you'll find branches 00111 10111 01000 and 11000 Similarly mark each of these "1/32".
Now the totality of the marked branches of the tree sum to exactly 1/2. So the answer is "five times".

7. If b and c are odd integers, prove that the quadratic equation

$$x^2 + 2bx + 2c = 0$$

cannot have rational roots.

Solution: Let $\frac{s}{t}$ be a rational root of the given equation, in lowest terms. Then

$$\frac{s^2}{t^2} + 2b\frac{s}{t} + 2c = 0.$$

Equivalently, $s^2 = -2bst - 2ct^2$.
It follows that s^2 is even. Therefore s is even.
Let $s = 2k$. We now have

$$4k^2 = -4bkt - 2ct^2$$

So $2k^2 + 2bkt = -ct^2$, which implies that ct^2 is even. Since c is odd, t^2—and so also t—must be even.
This contradicts the assumption that $\frac{s}{t}$ is in lowest terms. By contradiction, therefore, the equation cannot have a rational root.

Alternate #1: The discriminant of the given quadratic polynomial is $4b^2 - 8c = 4(b^2 - 2c)$. If the roots are rational (note either both are rational or both are irrational), this must be a perfect square. It would follow that $b^2 - 2c$ is a perfect square. Since b and c are odd, then $2c \equiv -2c \equiv 2 \pmod 4$ and $b^2 \equiv 1 \pmod 4$, so $b^2 - 2c \equiv 3 \pmod 4$ and so it cannot be a square (well-known and/or a straightforward check).
The result follows (again, by contradiction).

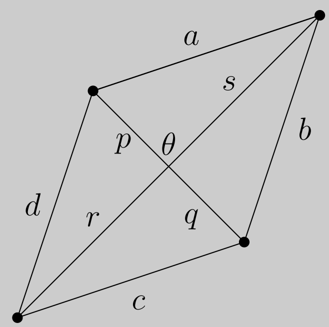
Alternate #2: (sketch) First show that, for any monic polynomial, any rational roots must be integers. Since $x^2 = -2bx - 2c$, x is even, and $x^2 + 2bx = -2c$. Since c is odd the right hand side is not divisible by 4, but, since x is even the left hand side is divisible by 4. This contradiction means there are no integral and therefore no rational roots.

Alternate #3: (sketch) In order to have rational roots, $4b^2 - 8c$ has to be a square of an integer, and the roots should be either integers or half-integers. Half integers are not possible, because x^2 is a fraction and $2bx + 2c$ is integer. So solutions are integers. x cannot be odd integer (in this case $x^2 + 2bx + 2c$ is odd), nor even (in this case $x^2 + 2bx$ is divisible by 4 and $2c$ is not).

8. A convex quadrilateral has sides of lengths a, b, c, d with the sides of length a and c opposite. Prove that, if $a^2 + c^2 = b^2 + d^2$, then the two diagonals of the quadrilateral are perpendicular.

Solution:

Let the diagonals intersect in segments of lengths p, q, r, s as shown, with angle θ as labelled.



By the law of cosines (and the fact that $\cos(180 - \theta) = -\cos\theta$) we have

$$a^2 + c^2 = p^2 + s^2 - 2ps\cos\theta + r^2 + q^2 - 2rq\cos\theta$$
$$b^2 + d^2 = p^2 + r^2 + 2pr\cos\theta + q^2 + s^2 + 2qs\cos\theta$$

Equating right hand sides gives

$$-2(ps + qr)\cos\theta = 2(pr + qs)\cos\theta.$$

So $(pr + qs + ps + qr)\cos\theta = 0$. Now $pr + qs + ps + qr > 0$ since $p, q, r, s, t > 0$. Therefore $\cos\theta = 0$ so $\theta = 90°$ and the diagonals are perpendicular as required.

9. How many solutions are there to the equation

$$x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + \cdots + x_{122}x_{123} + x_{123}^2 = 1$$

in integers $x_1, x_2, \ldots, x_{123}$? It should be clear from your work how you know that you have found all solutions.

Solution: Multiply the equation by 2 to get $2x_1^2+2x_1x_2+2x_2^2+2x_2x_3+\cdots+2x_{122}x_{123}+2x_{123}^2 = 2$ which we can rewrite as $x_1^2+(x_1+x_2)^2+(x_2+x_3)^2+\cdots+(x_{122}+x_{123})^2+x_{123}^2 = 2$. If the solutions are to be integers, we see that each of these squares must be the square of an integer. Clearly the only possibility is that exactly two of the squares are 1 and the rest are 0.

There are 124 squares. So choose two of them (can be done in $\frac{123\cdot 124}{2}$ ways). Let $y_1 = x_1$, $y_i = x_{i-1} + x_i$ for all $2 \leq i \leq 122$ and $y_{123} = x_{122}$. Once we have picked y_i and y_j with $i < j$ we can choose either $y_i = 1$ or -1. This forces $x_i = y_i$ and so $x_1 = x_2 = \cdots = x_{i-1} = 0$, $-x_i = x_{i+1} = -x_{i+2} = x_{i+3} = \cdots = (-1)^{j-i-1}x_j$, and $x_{j+1} = x_{j+2} = \cdots = x_{123} = 0$. So this determines all of the x_k. This means that there are $2\frac{123\cdot 124}{2} = 123 \cdot 124 = 15252$ solutions.

10. If α is a string of 0s and 1s let us denote by $\overline{\alpha}$ the string obtained by replacing every 0 with 1 and every 1 with 0. If β is another such string, let $\alpha\beta$ represent the string obtained by concatenating α and β. For example, if $\alpha = 011101$ and $\beta = 110110$ then $\alpha\overline{\beta} = 011101001001$.

    Define a sequence of strings as follows: $\alpha_0 = 0$ and for $n > 0$, $\alpha_n = \alpha_{n-1}\overline{\alpha_{n-1}}$. Thus $\alpha_1 = 01$ and $\alpha_2 = 0110$. Number positions in each string in the usual way: the 1st, 2nd and 3rd and 4th positions of α_2 are 0, 1, 1 and 0 respectively.

    Show that, if α_n has at least 2017 positions, then the symbol in the 2017th position is independent of n—and determine which symbol it is.

Solution: α_{n+1} is always twice the length of α_n. Therefore the length of α_n is 2^n, so whenever $2^n > 2017$—that is, $n \geq 11$, the string will have a 2017th position.

From $a_n = a_{n-1}\overline{a_{n-1}}$ if both sequences have an ith position, that is $i \leq 2^{n-1}$, then they match in this position. In other words there is a single infinite string $a_1a_2a_3\cdots$ such that for all n, $\alpha_n = a_1a_2\cdots a_{2^n}$. It follows that the position in question, a_{2017} in α_n, is independent of n for $n \geq 11$.
Further, $a_i = \overline{a_{i-2^n}}$ whenever $2^n < i \leq 2^{n+1}$. Thus

$$a_{2017} = \overline{a_{2017-1024}} = \overline{a_{993}} = a_{993-512} = a_{481} = \overline{a_{225}} = a_{97} = \overline{a_{33}} = a_1 = 0$$

Alternate #1: α_{n+1} is always twice the length of α_n. Therefore the length of α_n is 2^n, so whenever $2^n > 2017$—that is, $n \geq 11$, the string will have a 2017th position.

Since $\alpha_n = \alpha_{n-1}\overline{\alpha_{n-1}}$ the positions in α_n up to the 2^{n-1}th are the same in α_n and α_{n-1}. It follows (by a simple induction) that $\alpha_{11}, \alpha_{12}, \alpha_{13}, \ldots$ all have the same symbol in the 2017th position (independent of n) as required. By the same reasoning, the symbol in the mth position for any positive integer m will be constant, once there *is* an mth position.

Now, $\alpha_3 = 01101001$, and the first eight whole numbers in binary representation are 0, 1, 10, 11, 100, 101, 110 and 111; the symbol in the mth position, $m \leq 7$, in α_n for $n \geq 3$ is equal to 1 if the sum of the binary digits of m is odd, and 0 otherwise. If this is true for positions $m < 2^n$ then it must also be true for positions $m < 2^{n+1}$ since for $2^n \leq m < 2^{n+1}$, the binary representation of m is the same as that for $m - 2^n$ save the first position, which is a 1; so the parity of the sum of the digits of m is opposite that of $m - 2^n$, mirroring the relation of the symbols in the corresponding positions in $\alpha_n\overline{\alpha_n}$. Since $2017 = 1024 + 512 + 256 + 128 + 64 + 32 + 1$ the sum of its binary digits is 7, which is odd, the symbol in the 2017th position in α_n for $n \geq 11$ is therefore 0.