1. (a) Solve the equation \( x + \frac{6}{x+1} = 4 \).
(b) Solve the equation \( x^5 + 36x = 13x^3 \).

Solution: (a) Multiply both sides by \( x + 1 \):
\[
x(x + 1) + 6 = 4(x + 1), \quad x \neq -1
\]
\[
x^2 - 3x + 2 = (x - 1)(x - 2) = 0.
\]
Thus \( x = 1, 2 \) (both validated by substituting into original equation).
(b) Gather terms on left, simplify and factor:
\[
x^5 - 13x^3 + 36x = x(x^4 - 13x^2 + 36) = x(x^2 - 4)(x^2 - 9) = 0.
\]
Thus \( x = 0, \pm 2, \pm 3 \) (solutions validated by substituting into original equation).

Note: In both parts the validation step is unnecessary because the solution is extracted from a relation clearly equivalent to the original. (In (a), \( x = -1 \) is not a possible solution.)

2. (a) Find the real numbers \( a \) and \( b \) if 2 and 3 are roots of \( x^3 + ax^2 + bx + 6 = 0 \).
(b) In this problem \( A \) and \( B \) are the two points at which the graph of the equation \( x^2 + y^2 = 8 \) meets the graph of the equation \( y = |x| \). What is the length of the segment \( AB \)?

Solution: (a) Substitute 2 and 3 into the equation:
\[
8 + 4a + 2b + 6 = 0
\]
\[
27 + 9a + 3b + 6 = 0.
\]
Simplify:
\[
2a + b = -7 \quad (1)
\]
\[
3a + b = -11. \quad (2)
\]
Subtract (2) \(- (1)\) to obtain \( a = -4 \). Substituting into (1) gives \( (a, b) = (-4, 1) \).

Alternate: (a) The product of the roots is \(-6\) so the third root is \(-1\). Thus the polynomial is \((x - 2)(x - 3)(x + 1) = x^3 - 4x^2 + x + 6\), and \( (a, b) = (-4, 1) \).

(b) By the second condition \( y^2 = x^2 \), so \( 2x^2 = 8; x = \pm 2 \), \( y = 2 \). The points \( A \) and \( B \) are \((-2, 2)\) and \((2, 2)\), so the length of segment \( AB \) is \( 2 - (-2) = 4 \).
3. (a) Find an equation of the circle passing through the origin and the points with coordinates (10, 0) and (0, 8).

(b) Find an equation of the line tangent to the circle with equation \((x - 2)^2 + (y + 1)^2 = 25\) at the point with coordinates (5, 3).

Solution: (a) Since \(A(0,0), B(10,0)\) and \(C(0,8)\) are vertices of a right triangle the center of the circle is the midpoint, \(D(5,4)\), of the hypotenuse \(BC\) and its radius is \(|DC| = \sqrt{41}\). The required equation is therefore \((x - 5)^2 + (y - 4)^2 = 41\).

(b) The slope of the radius from the center \((2,-1)\) to the point of tangency \((5,3)\) is \(\frac{3+1}{5-2} = \frac{4}{3}\); the slope of the tangent line is thus \(-\left(\frac{4}{3}\right)^{-1} = -\frac{3}{4}\). The required equation, in point-slope form, is therefore \(y - 3 = -\frac{3}{4}(x - 5)\).

Alternate answers: (b) Slope-intercept form: \(y = -\frac{3}{4}x + \frac{27}{4}\)
Standard form: \(3x + 4y = 27\).

4. (a) In this problem \(c\) and \(d\) are real numbers. The point on the graph of the equation \(y = x^2 + cx + d\) which is nearest to the \(x\)-axis is \((-2,5)\). find the values of \(c\) and \(d\).

(b) Car \(A\) is travelling due west at a constant speed of 50 km/hr. Car \(B\) is travelling due east at a constant speed of 60 km/hr. At 1:00 p.m. car \(A\) is 40 km due north of car \(B\). At 2:00 p.m. what is the distance between the two cars (as the crow flies)?

Solution: (a) The equation is that of a vertically oriented parabola, and may be rewritten

\[ y = \left(x + \frac{c}{2}\right)^2 + d - \frac{c^2}{4}. \]

The point nearest the axis is clearly the vertex, so \((-\frac{c}{2}, d - \frac{c^2}{4}) = (-2,5)\). The two corresponding equations immediately yield \((c, d) = (4,9)\).

(b) In a standard coordinate system with the usual orientation, miles for units and \(B\) at the origin at 1:00 p.m, the coordinates of the cars at 2:00 p.m. are \(A(-50, 40)\) and \(B(60,0)\). From the distance formula (or Pythagoras: hypotenuse of a right triangle with sides 40 and 110) the required distance is \(\sqrt{40^2 + (50 + 60)^2} = 10\sqrt{137} \text{ km.}\) (Approximate value \(\approx 117.047 \text{ km.} — \text{not required.}\)
5. A fenced property has the shape of a rhombus, as in the figure. The length of each side of the rhombus is 20 m. A dog outside the property is tethered to one corner of the rhombus as shown in the diagram. If the dog’s leash is 30 m long, how large an area can the dog cover?

Solution: Sketching the boundary one obtains that the region described consists of \( \frac{2}{3} \) of a circle of radius 30 m and \( \frac{2}{3} \) of another circle radius 10 m (in two sectors each subtending 120°). Thus the total area is 
\[
\frac{2}{3} \cdot 2\pi \cdot 30^2 + \frac{2}{3} \cdot 2\pi \cdot 10^2 = \frac{2000\pi}{3} \text{ m}^2.
\]

6. A race track is built with two straight parallel sides and semicircles at the ends (as in the figure). The parallel sides are 100 m long and \( \frac{100}{\pi} \) m apart. Runner Alpha at position \( A \) starts running clockwise around the track at 2 m/sec. At this precise moment a second runner Beta enters the track at position \( B \) which is 100 m from position \( A \), running at 5 m/sec. If Beta wants to meet Alpha as soon as possible, should he run clockwise or counterclockwise around the track to achieve his goal?

Solution: The semicircles together equal the circumference of a circle of radius \( \frac{50}{\pi} \), or 100 m. Running clockwise: the distance to cover is 100 m at an effective speed of \( 5 - 2 = 3 \) m/sec, which will take \( \frac{100}{3} = \frac{200}{6} \) sec. Running counterclockwise: the distance to cover is \( 100 + 100 = 200 \) m at an effective speed of \( 5 + 2 = 7 \) m/sec, which will take \( \frac{200}{7} < \frac{200}{6} \) sec. Beta should run counterclockwise to achieve his goal.
7. For what values of \( x \) does \( \frac{1}{x+1} + \frac{1}{2x} > 1 \) hold?

Solution: Let \( f(x) = \frac{1}{x+1} + \frac{1}{2x} - 1 = \frac{2x^2 + x + 1}{2x(x+1)} = \frac{(2x+1)(1-x)}{2x(x+1)} \). Sign analysis:

<table>
<thead>
<tr>
<th>( x )</th>
<th>( -\infty, -1 )</th>
<th>( -1, -\frac{1}{2} )</th>
<th>( -\frac{1}{2}, 0 )</th>
<th>( 0, 1 )</th>
<th>( 1, \infty )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( 2x+1 )</td>
<td>-</td>
<td>-</td>
<td>+</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( 1-x )</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>+</td>
<td>-</td>
</tr>
<tr>
<td>( 2x )</td>
<td>-</td>
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<td>-</td>
<td>+</td>
<td>+</td>
</tr>
<tr>
<td>( x+1 )</td>
<td>-</td>
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</tr>
</tbody>
</table>

The inequality holds where \( f(x) > 0 \)—that is, for \(-1 < x < -\frac{1}{2} \) and \( 0 < x < 1 \).

Alternate analysis: \( f(-2) < 0 \) and the value of \( f(x) \) changes sign at \(-1, -\frac{1}{2}, 0, 1 \) (because these are poles/roots of odd degree); the solution is immediate.

8. In this problem \( x, y \) and \( z \) are real numbers. Find all possible values of \( a \) if:

\[
a = \frac{x}{|x|} + \frac{y}{|y|} + \frac{z}{|z|}.
\]

Solution: Each term is \( \pm 1 \), according as the variable involved is > 0 or < 0. Since the variables are independent, the possible sums are any number of the form \( a = p + q + r \), where \( p, q, r \in \{ \pm 1 \} \). That is, \( a \in \{-3,-1,1,3\} \).

9. Prove that, if \( a + b + c = 0 \), then \( a^3 + b^3 + c^3 = 3abc \).

Solution: Form the polynomial with roots \( a, b, c \): \( p(x) = x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc \).

Directly, we have

\[
p(a) + p(b) + p(c) = a^3 + b^3 + c^3 - (a + b + c)(a^2 + b^2 + c^2) + (ab + ac + bc)(a + b + c) - 3abc
= a^3 + b^3 + c^3 - 3abc = 0 + 0 + 0 = 0,
\]

and the conclusion follows immediately.

Alternate #1: Cube \( a + b + c \):

\[
0 = (a + b + c)^3
= a^3 + b^3 + c^3 + 3ab^2 + 3ca^2 + 3ab^2 + 3cb^2 + 3ac^2 + 3bc^2 + 6abc
= a^3 + b^3 + c^3 + 3ab(a + b) + 3ac(a + c) + 3bc(b + c) + 6abc
= a^3 + b^3 + c^3 - 3ab(c) - 3ac(b) - 3bc(a) + 6abc = a^3 + b^3 + c^3 - 3abc,
\]

and the result follows immediately.
Alternate #2: \( c = -(a + b) \), so
\[
a^3 + b^3 + c^3 = a^3 + b^3 - (a + b)^3
= (a + b)(a^2 - ab + b^2 - (a + b)^2)
= -c(-3ab) = 3abc
\]
and the result follows immediately.

10. In the diagram \( \triangle ABC \) is isosceles with \( AB = AC \). Prove that if \( LP = PM \), then \( LB = CM \).

\[
\text{Solution: Add point } D \text{ on } BC \text{ so that } LD \parallel AC. \text{ Then } \angle BDL = \angle PCA, \angle LDP = 180 - \angle BDL = 180 - \angle PCA = \angle PCM, \text{ and } \angle CPM = \angle DPL, \text{ so } \triangle DLP \cong \triangle CMP, \text{ by SAA. Further, } \triangle BDL \text{ is isosceles with base } BD, \text{ so } LB = LD = CM, \text{ as required.}
\]

Alternate solution: As in the first solution, \( \angle BPL = \angle CPM \), while \( \angle PBA \) and \( \angle PCM \) are supplementary. Since supplementary angles have the same sine, by the sine law we have
\[
\frac{|LB|}{\sin \angle BPL} = \frac{|LP|}{\sin \angle PBA} = \frac{|PM|}{\sin \angle PCA} = \frac{|PM|}{\sin \angle PCM} = \frac{|CM|}{\angle CPM} = \frac{|CM|}{\sin \angle BPL},
\]
and the result follows immediately.