2007 Manitoba Mathematical Contest

Solutions

1. a) If a is a real number such that $a - \frac{1}{a} = \frac{5}{6}$, find the numerical value of $a^2 + \frac{1}{a^2}$.

<u>*Solution:*</u> $\left(a - \frac{1}{a}\right)^2 = a^2 + \frac{1}{a^2} - 2 = \left(\frac{5}{6}\right)^2$, so $a^2 + \frac{1}{a^2} = \left(\frac{5}{6}\right)^2 + 2 = \frac{97}{36}$.

1. b) Solve the equation: $\frac{4}{x-1} - \frac{9}{x^2-1} = 4$.

<u>*Solution:*</u> Multiplying all terms on both sides by $x^2 - 1$ one obtains

$$\begin{aligned} 4(x+1) - 9 &= 4(x^2-1); \\ 4x^2 - 4x + 1 &= 0; \\ (2x-1)^2 &= 0, \end{aligned}$$

whose only solution is $x = \frac{1}{2}$, which is easily verified to satisfy the original equation.

NOTE: Technically it is necessary to verify that the solution is valid, although marking was lenient on this point.

2. a) Find the area of triangle ABC if $AC = BC = 6$ and $\angle ACB = 120°$.

<u>*Solution:*</u> Let D be the base of a perpendicular to AB from C, dividing the triangle into two 30-60-90 triangles. Thus $CD = 3$ and $AD = DB = \frac{3\sqrt{3}}{2}$. The area of $\triangle ABC$ is thus

$$\frac{1}{2}\left(2 \cdot \frac{3\sqrt{3}}{2}\right)(3) = 9\sqrt{3}.$$

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2. b) If $9\cos^2\theta = 6\cos\theta - 1$, find the numerical value of $\tan^2\theta$.

<u>*Solution:*</u> Rearranging the equation we obtain

$$0 = 9\cos^2\theta - 6\cos\theta + 1 = (3\cos\theta - 1)^2,$$

so that $\cos\theta = \frac{1}{3}$. Therefore,

$$\tan^2\theta = \frac{\sin^2\theta}{\cos^2\theta} = \frac{1-\cos^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta} - 1 = \frac{1}{(1/3)^2} - 1 = 9 - 1 = 8.$$

3. a) A straight line with slope -2 meets the positive x-axis at A and the positive y-axis at B. If the area of $\triangle AOB$ is 7, what is the equation of this line? (In this problem "O" denotes the origin.)

Solution: Let $A = (a, 0)$ and $B = (0, b)$. The slope of AB, then, is $\frac{b-0}{0-a} = -\frac{b}{a} = -2$, so $b = 2a$. The area of $\triangle AOB$ is $\frac{1}{2}ab = \frac{a}{2}(2a) = a^2 = 7$, so $a = \sqrt{7}$ and $b = 2\sqrt{7}$. In intercept form, then, the line is

$$\frac{x}{\sqrt{7}} + \frac{y}{2\sqrt{7}} = 1,$$

or in slope-intercept form,

$$y = -2x + 2\sqrt{7}.$$

3. b) Give an example of a quadratic equation whose roots are the squares of the roots of the equation $x^2 - 2x - 4 = 0$.

Solution: Let p, q be the roots of the equation. Then $x^2 - 2x - 4 = (x - p)(x - q) = x^2 - (p + q)x + pq$, from which we obtain $p + q = -2$ and $pq = -4$. Thus, $p^2q^2 = (-4)^2 = 16$ and $p^2 + q^2 = (p + q)^2 - 2pq = (-2)^2 - 2(-4) = 4 + 8 = 12$. Thus, p^2 and q^2 are roots of the quadratic equation

$$x^2 - (p^2 + q^2)x + p^2q^2 = 0;$$
$$x^2 - 12x + 16 = 0$$

NOTE: Another approach explicitly uses the two roots, $1 \pm \sqrt{5}$, of the original equation.

4. a) A rectangular box has faces with areas 14 cm^2, 20 cm^2 and 70 cm^2. Find the volume of this box.

Solution: Let the box have dimensions x, y and z. The product of the areas of the faces is $(xy)(xz)(yz) = 14 \cdot 20 \cdot 70 = (xyz)^2$, the square of the volume. Hence the box has volume $\sqrt{14 \cdot 20 \cdot 70} = \sqrt{7^2 \cdot 2^2 \cdot 10^2} = 7 \cdot 2 \cdot 10 = 140$.

4. b) A circle has its center at $(2, 1)$. The line whose equation is $3x - 4y + 8 = 0$ is a tangent to this circle. What is the area of this circle?

Solution: A standard formula for the distance from a point to a line gives the distance from $(2, 1)$ to the line $3x - 4y + 8 = 0$ as

$$\frac{|3(2) - 4(1) + 8|}{\sqrt{3^2 + 4^2}} = \frac{10}{5} = 2.$$

Since this is the radius of the circle, we calculate its area as $\pi(2)^2 = 4\pi$.

NOTE: A slower approach that doesn't require the formula for the distance of a point to a line first locates the point of intersection of the circle and the line by intersecting perpendicular lines. The radius is the distance from this point to $(2, 1)$.

5. a) INSERT DIAGRAM

In the diagram the line AB is parallel to the line DE. The line CB bisects $\angle FCE$ and the line CA bisects $\angle FCD$. Prove that F is the midpoint of the line segment AB.

Solution: We are given that $\angle DCA = \angle ACF$. Also, $\angle DCA = \angle CAF$, since they are opposite angles on a transversal to parallel lines DE and AB. It follows that $\angle ACF = \angle CAF$, so $\triangle ACF$ is isosceles, with equal sides AF and CF.

In the same way we argue that $\triangle CBF$ is iscosceles, with equal sides FB and CF. Therefore, $AF = CF = FB$, so F bisects AB.

5. b) The circle in the diagram has radius 1. The length of the chord AB is 1 and the length of the chord BC is $\sqrt{2}$. Find the length of the chord AC.

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<u>Solution:</u> Let the center of the circle be O. From the given information, $\triangle AOB$ is equilateral and $\triangle BOC$ is right-isosceles with $\angle BOC = 90°$. Thus $\angle AOC = \angle AOB + \angle BOC = 60 + 90 = 150°$. Applying the cosine law to $\triangle AOC$ we have

$$(AC)^2 = (AO)^2 + (CO)^2 - 2(AO)(CO)\cos 150 = 1^2 + 1^2 - 2(1)(1)\left(-\frac{\sqrt{3}}{2}\right) = 2 + \sqrt{3}.$$

Hence $AC = \sqrt{2 + \sqrt{3}}$.

NOTE: the answer can also be given as $\frac{\sqrt{2}+\sqrt{6}}{2}$ *– easily seen to have the same value. Another approach uses the sine law.*

6. If a and b are real numbers, what is the least possible value of $a(ab^2 + 3b) + 5$?

<u>Solution:</u> Taking $ab = u$ the expression may be rewritten as

$$(ab)^2 + 3ab + 5 = u^2 + 3u + 5 = \left(u + \frac{3}{2}\right)^2 + 5 - \left(\frac{3}{2}\right)^2.$$

Since squares cannot be negative, the minimum value is clearly reached if the square term involving u is 0, namely, when $u = -\frac{3}{2}$ (u attains this value, for example, when $a = -3$, $b = \frac{1}{2}$). The minimum value thus obtained is $0^2 + 5 - \left(\frac{3}{2}\right)^2 = \frac{11}{4}$.

Technically, the parenthesized point is necessary, but few students actually exhibited a way for u to take on the required value. Marking was lenient on this very fine point.

7. The point A is on the line whose equation is $y = 2x$, the point B is on the line whose equation is $y = -2x$ and the length of the line segment AB is 2. Prove that the coordinates of the midpoint of AB satisfy the equation $16x^2 + y^2 = 4$.

<u>Solution:</u> Take $A = (a, 2a)$ and $B = (b, -2b)$. Since the length of AB is 2, the distance formula gives

$$(a - b)^2 + (2a - (-2b))^2 = 5a^2 + 6ab + 5b^2 = 2^2 = 4.$$

The midpoint of AB is $(x, y) = \left(\frac{a+b}{2}, a - b\right)$. From the above result we have (as required),

$$16x^2 + y^2 = 16\left(\frac{a+b}{2}\right)^2 + (a - b)^2 = 5a^2 + 6ab + 5b^2 = 4.$$

8. Prove that, if two prime numbers differ by 2, and both numbers are greater than 3, then their sum is divisible by 12.

<u>Solution:</u> Let the two primes be $n \pm 1$. Clearly 3 divides one of $n-1, n, n+1$, and since $n - 1 > 3$, neither $n - 1$ nor $n + 1$ is divisible by 3; so n must be divisible by 3. Further, n is even, so we can take $n = 2 \cdot 3 \cdot m$. The sum of the two primes is $(n - 1) + (n + 1) = 2n = 12m$.

9. The equation of the circle in the diagram is $x^2 + y^2 = 25$. The chords AB and CD meet at P. The chord CD is parallel to the x-axis and has length 6. The chord AB has length 8 and $\angle BPD = 45°$. What are the coordinates of the point P?

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<u>*Solution:*</u> Since $\angle BPD = 45°$ the equation of line AB is $y = x + b$, where b is the y-intercept. Solving the system

$$\begin{aligned} x^2 + y^2 &= 25 \\ y &= x + b \end{aligned}$$

We obtain two solutions $(x, y) = \left(\frac{-b \pm \sqrt{50-b^2}}{2}, \frac{b \pm \sqrt{50-b^2}}{2}\right)$, which relate the coordinates of A and B to b. Since the length of segment AB is 8 we have

$$\begin{aligned} (AB)^2 &= \left(\frac{-b + \sqrt{50-b^2}}{2} - \frac{-b - \sqrt{50-b^2}}{2}\right)^2 + \left(\frac{b + \sqrt{50-b^2}}{2} - \frac{b - \sqrt{50-b^2}}{2}\right)^2 \\ &= 100 - 2b^2 = 64, \text{ giving } b = 3\sqrt{2}. \end{aligned}$$

Since D forms a right triangle with the origin and the y-intercept of CD, with hypotenuse 5 and one side 3, the y-intercept must be $(0, 4)$ and the equation of line CD must be $y = 4$. Solving $y = 4 = x + 2\sqrt{3}$ for x gives the coordinates of the intersection, P, of the two lines, namely $(4 - 3\sqrt{2}, 4)$.

10. a) Prove that a triangle in a rectangle of area A has area at most $\frac{A}{2}$.

<u>*Solution:*</u> Let X, Y, Z be any three points in the rectangle, and let L be the line through Z and parallel to line XY. If all four vertices of the rectangle lie on the same side of L as X and Y then the entire rectangle lies on the same side of L. It follows that Z is on L. Otherwise, one vertex of the rectangle, call it Z', lies strictly on the opposite side of L. Thus, the altitude to line XY to Z' is larger than the altitude to Z. It follows that the area of $\triangle XYZ'$ is strictly greater than the area of $\triangle XYZ$. Arguing similarly for points X and Y we reason that the area of $\triangle XYZ$ does not exceed the area of some triangle $\triangle X'Y'Z'$, all of whose vertices are vertices of the rectangle. The area of such a triangle is clearly 0 or $A/2$, and the conclusion follows.

NOTE: Many students appeared to believe that the points were given to be on the <u>perimeter</u> of the rectangle, or that it was obvious that the maximum is attained only when X, Y, Z are vertices of the rectangle. But, in fact, showing one of these assertions to be true is the most important part of the proof; once this is established, the result follows easily. There are other ways to arrive at this point, such as by scaling $\triangle XYZ$ until its vertices bump into the rectangle and argue that this does not decreae its area, or by sliding Z along the line parallel to XY until it bumps into the perimeter. Without this step, an answer would be worth at most 2 marks, depending on content.

10. b) Use part (a) to prove that among any nine points in a square of area 8, no three of which are colinear, some three are vertices of a triangle of area at most 1.

<u>*Solution:*</u> Divide the square into four smaller squares of area 2. By the pigeon-hole principle, some three points are in the same small square. By part (a), the area of the triangle formed by them has area at most $\frac{2}{2} = 1$.