2019 MANITOBA MATHEMATICAL COMPETITION

(for students in grade 12)

Tuesday, February 26, 2019, 9 AM – 11 AM

Sponsors: Manitoba Association of Mathematics Teachers; Winnipeg Actuaries' Club; University of Manitoba; Canadian Mathematical Society

☐ Mr.
☐ Mrs.
☐ Ms.

↑ Student's Name (print). Underline the Surname ↑

↑ Student's Signature ↑

↑ Student's School ↑

↑ Student's Home Street Address (or P. O. Box #) ↑

↑ City/Town Postal Code ↑

Instructions for participants: Before the contest begins, complete the above information. Put no personal identifying information on any other pages. You should have received 12 pages in total, including this page.

Answer each question on the page where it appears. If you run out of space below a question work may be continued on the backs of any question pages or either side of the last page, but not the front or back of this cover page—see instructions on the last page (page 23 of 24). All surfaces except this side of this cover page may be used for scratch calculations.

In your work on any question do not refer to work done on any other questions; they are marked independently.

No aids are permitted—no straight edges, compasses or other mechanical drawing devices, electronics (cell phones, electronic watches, translators, tablets, calculators etc).

This space may be used for scratch work, but do not continue solutions on this page—no credit will be given for work appearing here.

Question 1

(a) Find an integer n satisfying

$$\frac{3}{7} < \frac{n}{6} < \frac{4}{7}$$

(b) Will, Bob, Sara and Ida wrote integers on the blackboard.

- Bob's number was 15 less than Will's number.
- Sara's number was equal to the square of Bob's number.
- Ida's number was one-half of Sarah's number.

If Ida wrote the number 72, what possible numbers might Will have written?

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 2

(a) The head of a fish is 10 inches long; the tail is as long as the head plus one-half of the body; the body is as long as the head and tail together. How long is the fish?

(b) A man who is 2 m tall stands 4.5 m from a street light. His shadow is 3 m long. How high is the street light?

This space may be used for scratch work, or to continue solutions—**see instructions on page 23 of 24**

Question 3

(a) How many ordered triples of positive integers (a, b, c) can be formed so that $abc = 12$? For example, one such triple is $(a, b, c) = (2, 2, 3)$.

(b) Prove that there is no ten-digit prime number which uses every decimal digit exactly once.

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 4

Solve each of the following equations

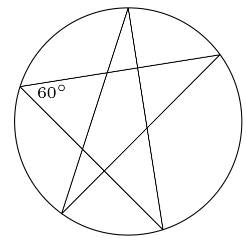
(a) $\dfrac{1}{x+1}+\dfrac{1}{x+3}=1$

(b) $(2x+y-3)^2+(x-2y+3)^2=0$

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 5

(a) Which number is greater, $A = \sqrt{6} + \sqrt{10}$ or $B = \sqrt{5} + \sqrt{12}$?

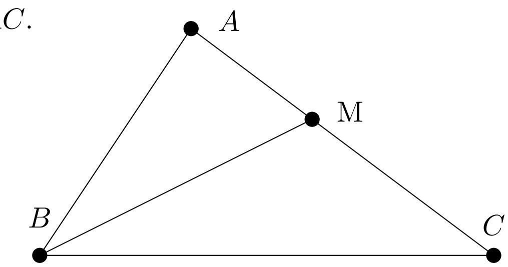


(b) Imogene draws an irregular five pointed star inside a circle. She measures the angles at the five points and finds that no two are equal. The largest one measures 60°. Show that the second largest is more than 30°.

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 6

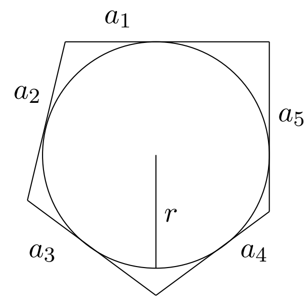
In triangle ABC, $\angle ABC$ is acute and M is a point on AC. If $\angle ABM = \angle CBM$ prove that $\frac{BA}{BC} = \frac{MA}{MC}$.



This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 7

A convex polygon with n sides has an inscribed circle of radius r, and side-lengths $a_1, a_2, \ldots, a_n$, in cyclic order.



(a) Suppose $n = 5$, and the sides of lengths a_1 and a_5 are perpendicular (shown). Find an expression for r in terms of a_1, a_2, a_3, a_4 and a_5.

(b) Suppose n is even. Prove that the alternating sum

$$a_1 - a_2 + a_3 - \cdots - a_n$$

is zero.

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 8

If $\sqrt{x-13}$ and $\sqrt{x+20}$ are both integers, find all possible values of x and prove that no other values are possible.

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 9

A child has 9 wooden blocks. Three of them have the letter A painted on each side. Three have the letter B and the other three have the letter C. She is asked to use 5 of these blocks to make a 5 letter sequence with the restriction that no two As can be consecutive. For example $CBBAC$ is an acceptable sequence but $ABAAC$ is not.

How many acceptable sequences are there?

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Question 10

Consider the equation

$$ax^2 + y^2 + 4xy + 2x + 2y = 0$$

where a, x, y are real numbers. For every real value of x there exists at least one real value of y. What are the possible values of a?

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24

Both sides of this page, and the backs of all question pages, are for scratch work or continuation of solutions.

To receive credit for work continued outside the page on which a question is written:

1. Clearly indicate in your solution that it is continued and specify where.
2. Where such work is continued, clearly indicate which question is being continued (e.g., "Q7 (cont.)").
3. Do not intermingle continued work from different questions or with scratch work.

This space may be used for scratch work, or to continue solutions—see instructions on page 23 of 24