2017 MANITOBA MATHEMATICAL CONTEST

For students in grade 12
9:00 AM – 11:00 AM
Tuesday, February 28, 2017

Manitoba Association of Mathematics Teachers

Winnipeg Actuaries' Club

Sponsored by:

The Winnipeg Actuaries' Club

The Manitoba Association of Mathematics Teachers

The Canadian Mathematical Society

The University of Manitoba

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Questions are found on both sides of this sheet. Answer as many as possible, but you are not expected to answer them all. **CALCULATORS ARE NOT PERMITTED**. Numerical answers by themselves, without explanation, will not receive full credit.

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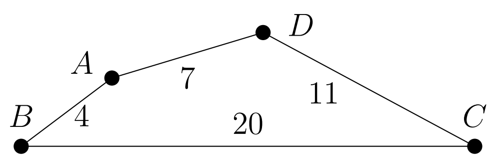
1. (a) In how many ways can \$20 be changed into dimes and quarters, if at least one of each type of coin must be used?

   (b) A box contains 5 nickels and 5 dimes. Two coins are selected at random. What is the probability that the total value of the selected coins is less than or equal to 15 cents?

2. (a) On a standard 8×8 chessboard one of the 64 unit squares is chosen at random. Determine the probability that the chosen square has no edge in common with the perimeter of the chessboard.

   (b) Determine the number of different two-letter words that can be formed using the letters of the words TRAP CARDS. (A two-letter word here consists of any pair of letters in a given order. A letter may be used twice only if it appears twice in the given phrase.)

3. (a) The sides of quadrilateral $ABCD$ are:
   $AB = 4$, $AD = 7$, $BC = 20$ and $DC = 11$.
   Given that AC is an integer, find its value.



   (b) Positive integers a, b, c and d are added three at a time. The sums are 93, 96, 105, and 114. What is the value of the largest of a, b, c and d?

4. (a) Suppose a rectangular sheet of paper, when folded in half and cut along the fold, forms two smaller sheets of paper with the same ratio of length to width as the original sheet. Find the ratio of length to width. Justify your answer.

   (b) If the original sheet of paper, described above, has an area of 1 m^2 we call it an A0 sheet of paper. It can be folded in half and cut to form two A1 sheets. These can be folded in half and cut to form A2 sheets. Continuing this process we generate A3 and A4 sheets of paper. The A4 sheet of paper is the standard office paper used in most countries of the world. The A4 sheet of paper has dimensions 2^a metres by 2^b metres. Find a and b.

5. (a) Both digits of a two digit number K (written in standard notation) are nonzero. When their order is reversed the new number is 45 less than K. Find all possible values for K.

   (b) N is an integer strictly between 10^3 and 10^4. Reversing the digits of N gives a number M also strictly between 10^3 and 10^4. Further, both M and N are divisible by 45. Find all possible values for N.

6. How many times must one flip a fair coin to guarantee that the probability of getting one or more runs of three consecutive heads or tails is 50% or more? Justify your answer.

7. If b and c are odd integers, prove that the quadratic equation

$$x^2 + 2bx + 2c = 0$$

   cannot have rational roots.

8. A convex quadrilateral has sides of lengths a, b, c, d with the sides of length a and c opposite. Prove that, if $a^2 + c^2 = b^2 + d^2$, then the two diagonals of the quadrilateral are perpendicular.

9. How many solutions are there to the equation

$$x_1^2 + x_1x_2 + x_2^2 + x_2x_3 + x_3^2 + \cdots + x_{122}x_{123} + x_{123}^2 = 1$$

   in integers $x_1, x_2, \ldots, x_{123}$? It should be clear from your work how you know that you have found all solutions.

10. If α is a string of 0s and 1s let us denote by $\overline{\alpha}$ the string obtained by replacing every 0 with 1 and every 1 with 0. If β is another such string, let $\alpha\beta$ represent the string obtained by concatenating α and β. For example, if $\alpha = 011101$ and $\beta = 110110$ then $\alpha\overline{\beta} = 011101001001$.

    Define a sequence of strings as follows: $\alpha_0 = 0$ and for $n > 0$, $\alpha_n = \alpha_{n-1}\overline{\alpha_{n-1}}$. Thus $\alpha_1 = 01$ and $\alpha_2 = 0110$. Number positions in each string in the usual way: the 1st, 2nd and 3rd and 4th positions of α_2 are 0, 1, 1 and 0 respectively.

    Show that, if α_n has at least 2017 positions, then the symbol in the 2017th position is independent of n—and determine which symbol it is.