2024 MANITOBA MATHEMATICAL COMPETITION

(for students in grade 12)

Wednesday February 28, 2024, 9 – 11 AM

Sponsors:

☐ Mr.
☐ Mrs.
☐ Ms.

↑ Student's Name (print). Underline the Surname ↑

↑ Student's Signature ↑

↑ Student's School ↑

↑ Student's Home Street Address (or P. O. Box #) ↑

↑ City/Town   Postal Code ↑

↑ Email ↑

Instructions for participants: Before the contest begins, complete the above information. Put no personal identifying information on any other pages. You should have received 12 pages in total, including this page, all showing the same paper number.

Answer each question on the page where it appears. If you run out of space below a question work may be continued on the back of the same page; if that is insufficient you may further continue work on blank pages 23 and 24.

No for-credit work should appear on the back of this cover page (it may be used for scrap).

Don't refer in any solution to work done on other questions; they are marked independently.

No aids are permitted—no straight edges, compasses or other mechanical drawing devices, electronics (cell phones, electronic watches, translators, tablets, calculators etc.).

This space may be used for scratch work. Do not continue solutions on this page—no credit will be given for work appearing here.

Question 1

(a) For how many values of y is the equation

$$\frac{y}{3+y} = \frac{1}{1+\frac{3}{y}}$$

a true statement?

(b) For how many real values of x is the equation

$$\frac{x+3}{x} = \frac{1}{1+\frac{x}{3}}$$

a true statement?

This space may be used to continue your solution for Question 1; it may be further continued on page 23 or 24—see instructions on page 23.

Question 2

(a) Find a positive integer n such that

$$\frac{1}{4n} - \frac{1}{4n+4} = \frac{1}{2024}.$$

(b) Is 2024 the sum of seven consecutive positive integers? Give a brief reason for your answer.

This space may be used to continue your solution for Question 2; it may be further continued on page 23 or 24—see instructions on page 23.

Question 3

Chris stuffs envelopes at a mail distribution center. Each morning Chris receives a pile of $2n$ envelopes to stuff that day. Chris has just stuffed exactly n envelopes at an average rate of 200 envelopes per hour when a memo arrives saying that anyone stuffing an average of at least 300 envelopes per hour that day will receive a \$100 bonus, and anyone stuffing an average of 400 envelopes per hour will receive a bonus of \$200 at the end of the day.

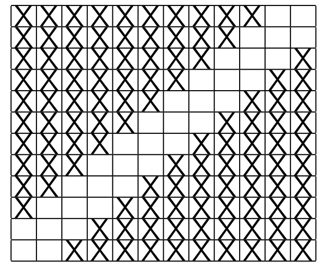
(a) At what rate must Chris stuff the remaining n envelopes in his pile in order to receive the \$100 bonus?

(b) At what rate must Chris stuff the remaining n envelopes in his pile in order to receive the \$200 bonus?

This space may be used to continue your solution for Question 3; it may be further continued on page 23 or 24—see instructions on page 23.

Question 4

A 12×12 grid has obstructions on every square except on three adjacent upwards diagonals as shown. Moving only to the right or upwards, one unobstructed square at a time, is the number of pathways from the lower left corner square A greater, less than, or equal to 2024?



Give a reason for your answer.

This space may be used to continue your solution for Question 4; it may be further continued on page 23 or 24—see instructions on page 23.

Question 5

Find the average of the six numbers a, b, c, d, e, f, where

$$\begin{aligned} 3a + 2b &= c \\ 3b + 2c &= d \\ 3c + 2d &= e \\ 3d + 2e &= f \\ 3e + 2f &= a \\ 3f + 2a &= b + 2024 \end{aligned}$$

This space may be used to continue your solution for Question 5; it may be further continued on page 23 or 24—see instructions on page 23.

Question 6

Suppose a, b, c are positive integers. Rewrite the expression

$$\frac{1}{a+b+c}\left[\frac{1}{a(a+b)}+\frac{1}{b(a+b)}+\frac{1}{a(a+c)}+\frac{1}{c(a+c)}+\frac{1}{b(b+c)}+\frac{1}{c(b+c)}\right]$$

as a single fraction in simplest terms.

This space may be used to continue your solution for Question 6; it may be further continued on page 23 or 24—see instructions on page 23.

Question 7

Consider the function (in which angles are measured in degrees)

$$f(x) = \frac{n + (5n+2)\sin^2(90n)}{2}.$$

Write $f^2(x) = f(f(x))$, $f^3(x) = f(f(f(x))) = f(f^2(x))$ and, more generally, for $n > 1$, $f^n(x) = f(f^{n-1}(x))$. Show that, for at least one positive integer n, $f^n(7) = 1$ and find the smallest such n.

This space may be used to continue your solution for Question 7; it may be further continued on page 23 or 24—see instructions on page 23.

Question 8

Show that there exist distinct positive integers $a, b, c, d > 1$ such that

$$d^{2024} - c^4 = c^4 - b^3 = b^3 - a^2,$$

and provide an example of such numbers.

This space may be used to continue your solution for Question 8; it may be further continued on page 23 or 24—see instructions on page 23.

Question 9

(a) Write 2024 in the form

$$\pm 3^{a_1} \pm 3^{a_2} \pm \cdots \pm 3^{a_t}$$

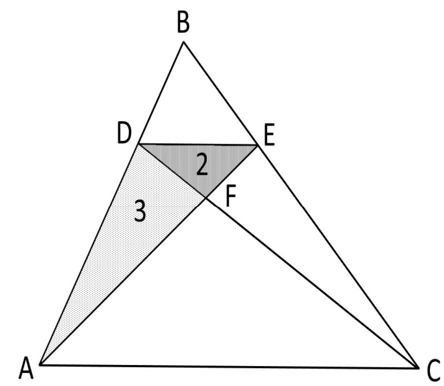
where $a_1, \ldots, a_t$ are distinct integers—that is, express 2024 as sum and/or difference of distinct integer powers of 3.

(b) Show that every integer n, positive or negative, can similarly be expressed as a sum/difference of distinct powers of 3 (in which the number of terms may be 0, or any positive integer.)

This space may be used to continue your solution for Question 9; it may be further continued on page 23 or 24—see instructions on page 23.

Question 10

In $\triangle ABC$ points D and E are chosen on the sides AB and BC so that DE is parallel to AC. Let F be the point of intersection of the lines AE and CD. Given that the area of $\triangle ADF$ is 3 and the area of $\triangle DEF$ is 2, find the area of $\triangle ABC$.



This space may be used to continue your solution for Question 10; it may be further continued on page 23 or 24—see instructions on page 23.

Both sides of this sheet may be used for continuation of solutions or for scratch work.

To receive credit for work continued here:

1. Clearly indicate in your solution that it is continued here.
2. Clearly indicate here which question is being continued (e.g., "Q7 (cont.)").
3. Clearly separate continued work from different questions and from scratch calculations.

This space may be used for scratch work, or to continue solutions—see instructions on page 23