2019 UMOMC

13$^{\text{th}}$ annual **University of Manitoba Open Math Challenge**

Are you a "mathlete"? Find out by testing your wits against these problems. This contest is open to undergrad students in any UM program. Participation costs you nothing but your time. You receive your results confidentially except that the overall winner receives a book prize and $100 cash, and fame. There is also a book prize for second place. Interested participants will receive feedback on their own work. For more details about UM Mathletics visit our web page: http://server.math.umanitoba.ca/~craigen/manitobamathletics/mathteam/

Instructions:

You are not expected to solve all questions; submit solutions only for those on which you have made significant progress; do not submit work you consider to be worthless. DO NOT BE DISCOURAGED by these questions—they are not intended to be "easy".

Whether or not a question explicitly states that a proof is required your solution should show reasoning and/or work demonstrating your answer to be complete and correct; in most questions a correct answer without justification will not receive full marks. Do not answer questions not asked; do not include scratch work or long explanations of how a solution is discovered, especially if this involves dead ends.

Submit well-presented solutions before 5:00 PM September 17 to R. Craigen (MH 523), D. Gunderson (MH 521) or V. Shepelska (MH 448) or, in a sealed envelope to the Math Office (MH 420), or by email to any of us from your UM email account before the deadline (typeset or high-quality scan of neat handwriting). Solutions will be judged according to correctness, completeness, clarity, elegance, and proper justification. Each question is worth 10 marks. Begin each solution on a new page. Staple solutions in the same order as questions are given.

HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult references or use technology to solve these problems.

1. Which integers can be expressed as the sum of three consecutive integers?

2. A sphere is spinning rapidly and an artist hits the sphere three times with paintballs. What is the probability that the exact centres of the three hits are contained in a single hemisphere?

3. Solve the following system of equations

$$\left\{ \begin{array}{ccccccc} x^2 & + & xy & + & xz & = & y \\ y^2 & + & yz & + & xy & = & z \\ z^2 & + & zx & + & zy & = & x \end{array} \right.$$

4. Let $a_1, a_2, \ldots, a_{2019}$ be real numbers satisfying

$$a_n + a_{n+1} = a_{n+1}^2 + 1$$

for all $1 \leq n \leq 2018$.

Given that $a_1 = a_{2019}$, find a_{1010}.

5. In $\triangle ABC$ points D and E are chosen on the sides AB and BC so that DE is parallel to AC. Let F be the point of intersection of the lines AE and CD. Given that the area of $\triangle ADF$ is 3 and the area of $\triangle DEF$ is 2, find the area of $\triangle ABC$.

6. Decide (with proof) whether or not $19^{19} = X^3 + Y^4$ has any integer solutions.

7. For a set of numbers X write $X + X$ for the set of numbers which are sums of two (not necessarily distinct) elements of X. Now suppose A, B, C are disjoint nonempty sets such that

   (a) $A + A \subseteq A$;

   (b) $B + B \subseteq C$; and

   (c) $C + C \subseteq B$.

   (d) $A \bigcup B \bigcup C = \mathbb{N}$ $(= \{0, 1, 2, 3, \ldots\})$

   Find (with proof) all possibilities for such a triple (A, B, C).

8. Let $f \geq 0$ be a continuous concave down function on $[0, 1]$ such that

$$\int_0^1 f(x)dx = 1.$$

   Prove that there is a rectangle of area at least $\frac{1}{4}$ contained in the region below the graph of $y = f(x)$ and above the x axis on this interval.

9. For some positive integer p, let $q = 2p$ and $A = (a_{ij})$ be a $p \times q$ matrix whose entries are -1, 0, or 1. Let $\mathbf{0}$ denote the column vector of q zeros. Decide if there is a nonzero solution $\mathbf{x} = (x_1, \ldots, x_q)^T$ to the matrix equation

$$A\mathbf{x} = \mathbf{0}$$

   such that for each $i = 1, \ldots, q$, x_i is an integer and $|x_i| \leq q$.

10. Two triangles $\triangle ABC$ and $\triangle DEF$ are similar with sides being integer lengths. Suppose that $|AB| = |DE|$ and $|AC| = |DF|$. If the remaining two sides BC and EF differ in length by 774, what are the possible lengths of the sides?