Mathletics Tools I

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"Never use a formula you can't prove". We can't all live by this rule, but let us strive toward it as a goal to be able to produce, on demand, proofs for every elementary formula we might use in a competition (forget about trying to prove the Four Colour Theorem and the Continuum Hypothesis...).

Also, know their names. And cite them properly when used.

Here are a few to start with.

1. State and prove Pythagoras' theorem.

2. Prove the converse of Pythagoras' theorem.

3. Let a, b be the sides adjacent to angle θ in a triangle, and c the opposite side. State and prove Pythagoras-like inequalities for the cases $\theta < \frac{\pi}{2}$ and $\theta > \frac{\pi}{2}$.

4. State and prove the cosine law.

5. State and prove the sine law.

6. Prove that $(u \cdot v)/(||u||\ ||v||) = \cos\theta$, where θ is the angle between u and v, for any vectors u and v in $\mathbb{R}^2$ or $\mathbb{R}^3$. Why is the result also valid in $\mathbb{R}^n$?

7. State and prove Cauchy's inequality (including the case of equality).

8. State and prove the AM/GM inequality (including the case of equality).

9. State and prove the triangle inequality for vectors in $\mathbb{R}^n$. What can be said about the case of equality?

10. Prove that if f is a polynomial and a, b, n are integers such that $a \equiv b \bmod n$, then $f(a) \equiv f(b) \bmod n$.

11. Prove the rearrangement inequality:

    If $a_1 \leq a_2 \leq \cdots \leq a_n$, $b_1 \leq b_2 \leq \cdots \leq b_n$, and $c_1, c_2, \ldots, c_n$ is some rearrangement of $b_1, b_2, \ldots, b_n$ (preserving multiplicity), then

    $$\sum_{k=1}^{n} a_k b_{n-k} \leq \sum_{k=1}^{n} a_k c_k \leq \sum_{k=1}^{n} a_k b_k.$$

    That is,

    $$a_1b_n+a_2b_{n-1}+\cdots+a_nb_1 \leq a_1c_1+a_2c_2+\cdots+a_nc_n \leq a_1b_1+a_2b_2+\cdots+a_nb_n.$$

Jensen's inequality: A multi-purpose inequalities tool