2015 Manitoba Mathematical Competition Key

Thursday, February 24, 2015

1. (a) Solve for x:

$$\frac{5}{x} - \frac{7+2x}{3x} = 3$$

(b) Find the ratio of x to y, given that

$$\frac{x-y}{x+y} = \frac{2}{5}$$

Solution: (a) Multiplying through by $3x$ gives $15 - 7 - 2x = 9x$, so $11x = 8$, or $x = \frac{8}{11}$.
(b) Clearing denominators gives $5(x-y) = 2(x+y)$, which simplifies to $3x = 7y$. If $y = 0$ then $x = 0$ but in this case the left hand side of the given expression is undefined, and so $y \neq 0$. It follows that $\frac{x}{y} = \frac{7}{3}$ (or $x : y = 7 : 3$).

Notes:

- For (b) if one is familiar with homogeneous expressions one can avoid handling of the extraneous solution $x = y = 0$ (leading to an undefined ratio, which may be puzzling to some students) by the following argument: The problem (both givens and required) are homogeneous in x, y so without loss of generality we rescale so that $x - y = 2$ and $x + y = 5$; adding gives $2x = 7$, subtracting gives $2y = 3$ and the result follows by dividing these.
- Students who arrived at the correct answer despite awkward handling of the extraneous solution received full marks. Solutions in fractional form or for $y : x$ instead of $x : y$ (as the question, as worded, requires) were also accepted. However, "$3x = 7y$" is an incomplete solution as it does not answer the question. Ditto for a conclusion that does not mention x and y, such as "$\frac{7}{3}$" or "$7 : 3$"
- A common error in (a) was to write $15 - (7 + 2x) = 15 - 7 + 2x$.

2. (a) Let "$*$" denote an operation defined as follows: $a * b = 2ab - b^3$.
Find the value of $(4 * 3) * 2$.

(b) The original price of an item is reduced by 20%. A month later its reduced price is doubled. At the end of the year it is sold in a discount sale. What exact percentage discount was applied if the sale price was equal to the original price?

Solution: (a) $(4 * 3) * 2 = 2(4 * 3) \cdot 2 - 2^3 = 4(2 \cdot 4 \cdot 3 - 3^3) - 8 = 4 \cdot (-3) - 8 = -20$.
(b) $(1 - .2) \cdot 2 \cdot x = 1$ gives $x = \frac{5}{8} = 1 - \frac{3}{8}$, so this is a reduction by $100 \times \frac{3}{8} = 37.5\%$.

Notes:

- In (b) the solution could be made more explicit by letting the original price be y and x the fraction applied in the final discount. Then $y(1 - 0.2) \cdot 2x = y$, which gives $x = \frac{5}{8}$. Many students left this as the answer, instead of the actual discount, which is $1 - x = \frac{3}{8} = 37.5\%$.

3. In a **Fibonacci-like sequence** each term is the sum of the two terms immediately preceding it. For example the Fibonacci sequence $1, 1, 2, 3, 5, 8, \ldots$ and the Lucas sequence $1, 3, 4, 7, 11, 18, \ldots$ are both Fibonacci-like sequences.

   (a) The first term of a Fibonacci-like sequence is 4 and the fifth term is 2. Find the second, third and fourth terms.

   (b) If $x, y, 2x - 1, x + 4$ is a Fibonacci-like sequence, find the values of x and y.

Solution:

(a) Let the second term be x. The sequence is $4, x, 4 + x, 4 + 2x, 8 + 3x$. Solving $8 + 3x = 2$ we get $x = -2$. The required terms are $-2, 2$, and 0.

(b) $2x - 1 = x + y$ and $x + 4 = y + (2x - 1)$. Solve to get $x = 3, y = 2$.

Notes:

- No assumption was made that students will have seen Fibonacci sequence or general linear recurrences, but that may have given a slight speed advantage for those who had. We expected students to work directly from the information given in the question; no special knowledge was required. This tests students' ability to read a relatively complex description of numerical relations and from it derive meaningful conclusions and formulae.

4. (a) Find, with justification, all pairs of real numbers (a, b) such that $a^2 + 2ab + b^2 = 9$ and $a - b = 5$.

   (b) Find, with justification, all pairs of real numbers (a, b) such that $a^2 - ab + b^2 = 0$.

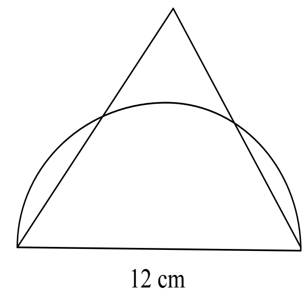
Solution: (a) $(a+b)^2 = 3^2$ so $a + b = \pm 3$. Solving with $a - b = 5$ by adding and subtracting gives $(a, b) = (1, -4)$ or $(4, -1)$.
(alternative for those who don't see the square: $(a-b)^2 = a^2 - 2ab + b^2 = 25$. Subtracting gives $4ab = 9 - 25 = -16$ so $a(-b) = 4$. Thus a and $-b$ are the two roots of the quadratic equation $t^2 - 5t + 4 = (t-4)(t-1) = 0$ So $\{a, -b\} = \{1, 4\}$. Thus, $(a, b) = (1, -4)$ or $(4, -1)$.)
(b) If $a^2 - ab + b^2 = 0$ then $(a+b)(a^2 - ab + b^2) = a^3 + b^3 = 0$, so $a^3 = -b^3$. Taking (uniquely determined) cube roots of both sides we have $a = -b$. Substituting into the relation gives $3a^2 = 0$, so $a = b = 0$. Since this is clearly a solution to the relation, the solution is $(a, b) = (0, 0)$.
Alternative for (b): consider this to be a quadratic equation in a, the discriminant is $-3b^2$ which is always non positive. Thus real solutions of a exist only if $b = 0$. The result follows immediately.

Notes:

- One clever student solution for (b): Complete the square to obtain $(a-b)^2 = -ab$, and conclude that $ab \leq 0$. But also $a^2 - ab + b^2 = 0$ gives $a^2 + b^2 = ab$, so $ab \geq 0$. It follows that $ab = 0$, so $a = 0$ or $b = 0$. Plugging either of these into the given relation forces the other to be 0. So $a = b = 0$.
- Another student solution to (b): Taking $c^2 = ab$, we have that $|a|, |b|, |c|$ are sides of a right triangle. As above, $ab \geq 0$ so a, b have the same sign. If $a, b > 0$ then $c > a$, $c > b$ so $c^2 > ab$—a contradiction, and similarly for $a, b < 0$. Thus $a = b = 0$.
- Yet another solution, also completing the square: $a^2 - 2(\frac{1}{2}b)a + \frac{1}{4}b^2 + \frac{3}{4}b^2 = 0$, or $(a - \frac{1}{2}b)^2 + \frac{3}{4}b^2 = 0$. A sum of squares equal to zero implies both terms are 0, so $a - \frac{1}{2}b = b = 0$, which implies $a = b = 0$.

5. (a) Find all possible values of a such that the line with equation $x + y = a$ is tangent to the circle with equation $x^2 + y^2 = 25$.

(b) One side of an equilateral triangle is the 12 cm diameter of a semi-circle, as shown. Find the area of the portion of the triangle that is outside the semi-circle.



Solution:

(a) A line tangent to a circle is perpendicular to a radial line to the same point. $x+y = a$ has slope -1. Radial lines pass through the origin which is the center of the circle, so the only candidate is $y = x$, which meets the circle at $\pm(\frac{5}{\sqrt{2}}, \frac{5}{\sqrt{2}})$. Plugging into the equation gives $a = \pm\frac{10}{\sqrt{2}} = \pm 5\sqrt{2}$.

Alternative #1: Squaring the first equation and subtracting the second gives $2xy = a^2 - 25$ so $xy = \frac{a-25}{2}$. From the sum and product of x, y the coordinates of any point of intersection are thus the roots of the quadratic $t^2 - at + \frac{a-25}{2}$. Tangency corresponds to having exactly one solution, so we set the discriminant, $a^2 - 2(a-25)$, to 0 and arrive at $a = \pm 5\sqrt{2}$.

Alternative #2: The x-axis, the y-axis and the line $x + y = a$ form a right isosceles triangle with legs of length $|a|$. The altitude to the hypotenuse is a radius of length 5. Thus $|a| = 5\sqrt{2}$ so $a = \pm 5\sqrt{2}$.

(b) Two lines from the center of the diameter to points of intersection form 60 degree angles (consider a hexagon inscribed in the full circle). Subtract area of 60 degree pie-slice of radius 6 from parallelogram consisting of two equilateral triangles of side 6: $18\sqrt{3} - \frac{36\pi}{6} = 6(3\sqrt{3} - \pi)$.

Notes:

- Variation on solution for (a): Substitute $y = a - x$ into the equation for the circle to get $(x-a)^2 + x^2 = 25$. Since tangency occurs the discriminant of this quadratic is 0. The discriminant simplifies to $200 - 4a^2$. Thus $a^2 = 50$ giving $a = \pm 5\sqrt{2}$.
- Some might be unfamiliar with the trick used in (a) Alternative Solution #1 to quickly find x and y. suppose you know the values of $x + y$ and xy. You could solve for one and plug it into the other equation, getting a quadratic equation to solve. Or you can take this shortcut: think of them as roots of a quadratic polynomial equation in variable t. Then $(t - x)(t - y) = t^2 - (x + y)t + xy = 0$. Knowing this, you can write down that polynomial in one step, saving time.

6. The sides of a right triangle are $x+2$, $x+6$ and $2x$. Find all possible values of x.

Solution: Since $x+2 < x+6$ the hypotenuse must be either $2x$ or $x+6$.

Case 1:

$$\begin{aligned}(x+2)^2+(2x)^2 &= (x+6)^2 \\ 5x^2+4x+4 &= x^2+12x+36 \\ 4x^2-8x-32 &= 0 \\ 4(x-4)(x+2) &= 0\end{aligned}$$

Since sides must have a positive length $x \neq -2$. Therefore $x = 4$.

Case 2:

$$\begin{aligned}(x+2)^2+(x+6)^2 &= (2x)^2 \\ 2x^2+16x+40 &= 4x^2 \\ 2x^2-16x-40 &= 0 \\ 2(x-10)(x+2) &= 0\end{aligned}$$

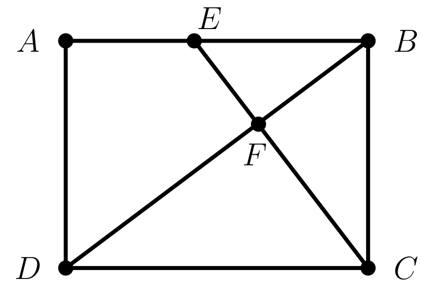
Again, $x \neq -2$, so $x = 10$.

The possible values of x are 4 and 10.

Notes:

- We saw a lot of algebraic difficulties. Particularly worrying were a number of cases where we observed incorrect squaring of binomials, with fatal errors like "$(x+2)^2 = x^2+4$".
- Many students neglected to explicitly eliminate the inadmissible value $x = -2$.

7. $ABCD$ is a rectangle. E is on AB. CE intersects diagonal DB at F. $\triangle ADE$ has area 50. $\triangle EFB$ has area 40. Find the area of rectangle $ABCD$.



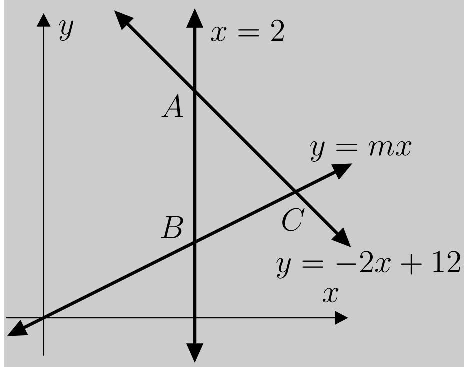
Solution: $\triangle CDE$ has the same area as $\triangle BCD$ (common base, same height). It follows that $\triangle DEF$ has the same area as $\triangle BFC$ (subtract the common triangle $\triangle CDF$). Since diagonal BD bisects the area of the rectangle it follows that the area of $\triangle CDF$ is equal to the sum of the areas of triangles $\triangle ADE$ and $\triangle BEF$. So $\triangle DFC$ has area $40+50=90$ giving us two similar triangles with area in the ratio $90:40=9:4$, so their sides are in the ratio $3:2$. Let $x=BE$. Then $CD=\frac{3}{2}x$. The respective heights of the triangles on these bases are $\frac{80}{x}$ and $\frac{120}{x}$, found by dividing area by half the base. The rectangle is therefore $\frac{3x}{2}\times\frac{200}{x}$ and so has area 300.

Notes:

- An intriguing aspect of the solution appears to be that one cannot solve explicitly for the dimensions of the rectangle.
- Few students solved this problem without introducing a lot of variables and algebra. In most cases this clutter got in the way and the student would get bogged down, but some managed to hack their way through the undergrowth to a successful solution (whew!). (Marks in such cases depended on whether a meaningful process was communicated to the marker. In the second half of the MMC a correct solution is expected to be accompanied by well-presented work that "makes the case" for the answer, so a correct answer cannot simply be accompanied by uninterpretable scratch work and receive much credit.)
- More on the introduction of variables: If you're introducing variables that are cancelled in the end, it should lead you to suspect that the variables are merely red herrings, and that a more elegant solution exists which does not need them.

8. The lines $y = -2x + 12$, $x = 2$ and $y = mx$ intersect to form a triangle with area 2. Find all possible values for m.

Solution:



Solving $y = -2x + 12$ with $x = 2$ gives point $A(2, 8)$. Solving $x = 2$ with $y = mx$ gives point $B(2, 2m)$. The x-coordinate of C is found by solving $mx = -2x + 12$, giving $x = \frac{12}{m+2}$, so the area of $\triangle ABC$ is
$\frac{1}{2}AB\cdot$(distance from C to AB)
$= \frac{1}{2}|8 - 2m| \cdot \left|\frac{12}{m+2} - 2\right|$
$\pm(4 - m)\left(\frac{8-2m}{m+2}\right) = 2$,
so we have two cases:

(i) $(4 - m)(8 - 2m) = 2m + 4$ so $(4 - m)^2 = m + 2 = 16 - 8m + m^2$ and therefore $m^2 - 9m + 14 = 0 = (m - 2)(m - 7)$; or

(ii) $(4 - m)(8 - 2m) = -2m - 4$, so $(4 - m)^2 = -(m + 1)$, giving $m^2 - 6m + 20 = 0$, which has negative discriminant and so has no real solutions—reject this case.

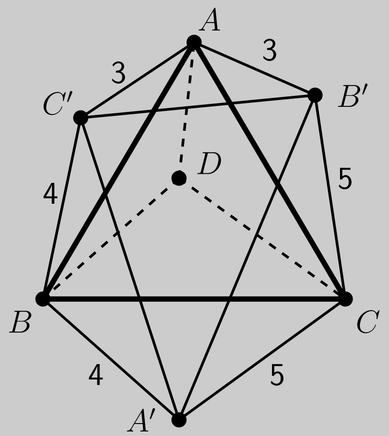
So $m = 2$ or $m = 7$.

Notes:

- The second case must be considered, and rejected.

9. Let A, B, C be vertices of an equilateral triangle, with D an interior point such that the lengths of AD, BD, CD are 3, 4 and 5, respectively. What is the area of the triangle?

Solution:



Reflect D through AB, BC and CA, to get points C', A' and B', respectively. Hexagon $AC'BA'CB'$ has twice the triangle's area with $120°$ angles at A, B and C. Triangles $\triangle C'BA'$, $\triangle A'CB'$, $\triangle B'AC'$ are 30-30-120 with short sides of lengths 4, 5 and 3 respectively, so $\triangle A'B'C'$ has sides $3\sqrt{3}$, $4\sqrt{3}$, $5\sqrt{3}$ and so is a right triangle. These four triangles partition the hexagon so its area is $\frac{3^2\sqrt{3}}{4} + \frac{4^2\sqrt{3}}{4} + \frac{5^2\sqrt{3}}{4} + \frac{1}{2} \cdot 3\sqrt{3} \cdot 4\sqrt{3}$ $= \frac{25\sqrt{3}}{2} + 18$. The area of $\triangle ABC$ is therefore $9 + \frac{25\sqrt{3}}{4}$.

Alternate: rotate $ABCD$ thru 60 degrees about A, so $A \to A$, $D \to D'$, $C \to C' = B$, $B \to B'$. Then $ADD' = 60°$, and $BDD' = 90°$, so $ADB = 150°$. The cosine law on ADB then gives $AB = \sqrt{25 + 12\sqrt{3}}$. So the area of $\triangle ABC$ is $\frac{\sqrt{3}}{4}\left(\sqrt{25+12\sqrt{3}}\right)^2 = \frac{25\sqrt{3}}{4} + 9$.

Alternate #2: Let $x = |AB| = |BC| = |CA|$. Using the cosine law, set $y = \cos(\angle DCB) = \frac{x^2+9}{10x}$ and $z = \cos(\angle DCA) = \frac{x^2+16}{10x}$. Sum-of-angles for cosine gives $\frac{1}{2} = \cos(\angle DAB + \angle DAC) = yz - \sqrt{1-y^2}\sqrt{1-z^2}$, so $(\frac{1}{2} - yz)^2 = (1-y^2)(1-z^2)$, which simplified to $\frac{3}{4} = y^2 + z^2 - yz$, a single equation in x, which simplifies to $x^4 - 50x^2 + 193 = 0$. Eliminating the extraneous root (too small) we obtain $x^2 = 25 + 12\sqrt{3}$ so the required area is $\frac{\sqrt{3}}{4}x^2 = 9 + \frac{25\sqrt{3}}{4}$.

Notes:

- Few solved this, although some made some progress toward extracting intermediate information. Most students had difficulty applying the given information in a useful manner. Some attempts to use Heron's formula got partway but ended up mired in a mess.

10. The expression $n!$ denotes the product $1 \cdot 2 \cdot 3 \cdots n$ and is read as "n factorial". For example $5! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 = 120$.

(a) The product $(2!)(3!)(4!)(5!)(6!)(7!)(8!)(9!)(10!)(11!)(12!)$ can be written in the form $M^2N!$ where M, N are positive integers. Find a suitable value of N and justify your answer.

(b) Prove that, for every $n \geq 1$, $(2!)(3!)(4!)\cdots((4n)!)$ can be written as the product of a square and a factorial.

Solution: (a)

$$2!3!\cdots 11!12! = \frac{2!^2}{2}\frac{4!^2}{4}\cdots\frac{10!^2}{10}\cdot 11!12! = (2!4!\cdots 10!)^2\frac{11!12!}{2\cdot 4\cdots 10}$$
$$= (2!4!\cdots 10!)^2(3^2 5^2\cdots 11^2)(2\cdot 4\cdot 6\cdots 12) = (2!4!\cdots 10!)^2(3\cdot 5\cdots 11)^2(2^6)6!$$

$= M^2N!$ where $M = 2!\cdot 4!\cdots 10!\cdot 3\cdot 5\cdots 11\cdot 2^3$ and $N = 6$.

(b) As in our solution to part (a), we have

$$2!\cdot 3!\cdots(4n)! = (2!\cdot 4!\cdots(4n-2)!)^2(3\cdot 5\cdots(4n-1))^2\cdot 2^{2n}(2n)! = M^2N!$$

where $M = 2^n 3!5!\cdots(2n-1)!$ and $N = 2n$.

Notes:

- For (a) there is a prime-factorization route $(2^{56}3^{26}5^{11}7^6 11^2)$, which is tedious but quite doable. One can eliminate $N > 12$ by observing that $13|n!$ for $n > 12$. $N \geq 5$ else the square is impossible, and $5!$ leaves only 2 and 3 with odd powers, so $N = 6$ works. This solution, however, doesn't provide much of a start for solving part (b).
- A cleaner way to do the bookkeeping for (a) is to write $p = 2!3!\cdots 12!$ and $q = 3!5!\cdots 11!$. Then $p/q^2 = 2\frac{4!}{3!}\frac{6!}{5!}\cdots\frac{12!}{11!} = 2\cdot 4\cdot 6\cdots 12 = 2^6\cdot 6!$, so it is clear that $N = 6$ works.

  For (b) we work similarly, with $p = 2!3!\cdots(4n)!$ and $q = 3!5!\cdots(4n-1)!$.