2014 Manitoba Mathematical Competition Key

Thursday, February 20, 2014

1. (a) In a picket fence each picket is 10 cm wide and the spaces between consecutive pickets are each 5 cm wide. What is the length of a straight fence if it has exactly 50 pickets?

   (b) Two kinds of pickets are available. Some are 10 cm wide and some are 15 cm wide. The spaces between pickets are 5 cm. Wide and narrow pickets alternate. What is the longest fence that can be built using 27 pickets?

Solution:

(a) Fifty pickets require 49 spaces. The length is $50(10) + 49(5) = 745$ cm.

(b) To make the fence as long as possible start and end with a wide picket. We have 14 wide pickets, 13 narrow pickets and 26 spaces. The length is $14(15) + 13(10) + 26(5) = 470$ cm.

Notes:

We're seeing increasingly many arithmetic errors on this type of question. Some students drew beautiful pictures. Some forgot to count inclusively or added an extra space at the end. Some, in part (b), counted by groups of two pickets and a space but neglected to account for the space between groups. Also in (b) many students performed complete calculations in two cases, one in which the first and last picket are wide and one in which they are narrow. This is an unnecessary waste of time; it is good for students to practice economy of thought. Contest questions on are often designed to reward students timewise for choosing more efficient approaches to problems.

2. In $\triangle ABC$, $AB = 3$, $BC = 4$ and $AC = 5$.

   (a) Find the length of the altitude from AC to B.

   (b) That same altitude divides $\triangle ABC$ into two smaller triangles. What is the ratio of their areas?

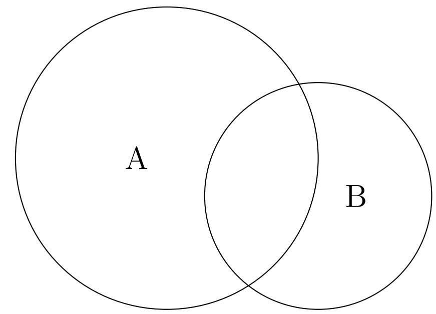
Solution:

(a) $(AB)^2 + (BC)^2 = 9 + 16 = 25 = (AC)^2$. Therefore $\angle ABC$ is a right angle. The area of the triangle is $\frac{1}{2}(AB)(BC) = \frac{1}{2}(3)(4) = 6$. If the altitude from B to AC is h, then $\frac{1}{2}(5)(h) = 6$, so $h = \frac{12}{5}$.
(Alternative)
$\triangle ABC$ is a right triangle with right angle at B. Let D be foot of altitude from AC to B. $\triangle BDC \sim \triangle ABC$, so $BD : BC = AB : AC$ so the altitude is $BD = \frac{(AB)(BC)}{AC} = \frac{3 \cdot 4}{5} = \frac{12}{5}$.

(b) They are similar (observe angles) right triangles whose hypotenuses (hypoteni?) have ratio $3 : 4$ so their areas have ratio $9 : 16$.

Notes:

The direction of the ratio is not clearly specified, so $16 : 9$ is just as acceptable. Fraction form was accepted as well. (Small) partial credit for a correct arbument in to (b) based on an incorrect answer in (a). A few justified the first conclusion by "Pythagoras' Theorem" but this is not strictly the case (no penalty though) – it is the *converse* that is being used, which is a distinct result.



3. (a) In the diagram, region A is $\frac{4}{5}$ of the large circle and region B is $\frac{5}{7}$ of the smaller circle. What is the ratio of areas A and B?

(b) There are 100 students in our school of which 50 take chemistry and 40 take physics. If there are 30 students taking both physics and chemistry, how many of these 100 students take neither physics nor chemistry?

Solution: (a) Labelling the intersection of the two circles C, we have $A : C = \frac{4}{5} : \frac{1}{5} = 4 : 1 = 8 : 2$ and $B : C = \frac{5}{7} : \frac{2}{7} = 5 : 2$, so $A : C : B = 8 : 2 : 5$ and so $A : B = 8 : 5$, or $\frac{8}{5}$.

(b) $100 - (50 + 40) + 30 = 40$ students.

Notes:

In contrast, a direction for the ratio is implied in part (a) here and a mark was lost by those reversing it unless they explicitly indicated so, as in writing "$B : A$" or "B/A". Many students misread the problem, taking A and B to be the areas of the two circles. Introducing radii and attempting to do all in terms of the formula for the area of circles led to serious wastes of time, confusion and many kinds of errors. Students should perceive that, regardless of the visual artifact, this is not a geometry question. In (b) a Venn diagram is appropriate (the part (a) diagram was a subtle hint) but those simply drawing one and filling in numbers but failing to answer the question lost a mark.

Simple principle: always answer a question in the same terms in which it was asked. (Which includes appropriate units, where relevant.) Answers should not include symbols or concepts not used in, or implied by, the question. For example, if a question does not use the symbol "x" one's answer should not be "$x = 3$", etc.

4. (a) The average value of 2013 consecutive integers is 2014. What is the smallest of these integers?

   (b) Let $a_1 = 2$, $a_2 = 2a_1$, $a_3 = 2a_2$ and so on. That is, for $n > 1$, $a_n = 2a_{n-1}$. Find and simplify the value of

$$\sqrt[2014]{a_1 \cdot a_2 \cdot a_3 \cdots a_{2014}} \cdot \sqrt{2}.$$

Solution:

(a) Pairing numbers in the obvious way we see that the middle number in the set is 2014. There are $\frac{2013-1}{2} = 1006$ numbers prior to it in the given set. Therefore the smallest of these integers is $2014 - 1006 = 1008$.

(b) We have $a_2 = 2^2$, $a_3 = 2^3$, $a_4 = 2^4$ and so on. That is, $a_n = 2^n$. So

$$\begin{aligned}\sqrt[2014]{a_1 \cdot a_2 \cdot a_3 \cdots a_{2014}} \cdot \sqrt{2} &= \sqrt[2014]{\left(2^1 2^2 2^3 2^4 \cdots 2^{2014}\right)} \cdot 2^{\frac{1}{2}} \\ &= \left(2^{1+2+3+\cdots+2014}\right)^{\frac{1}{2014}} \cdot 2^{\frac{1}{2}} \\ &= 2^{\frac{2014 \cdot 2015}{2 \cdot 2014} + \frac{1}{2}} = 2^{\frac{2015+1}{2}},\end{aligned}$$

which simplifies to 2^{1008}.

Notes:

One common inefficient approach in (a) was to start by working out 2013×2014 in order to get the sum of the numbers. Similarly one can begin with the expression

$$\frac{n_1 + (n_1 + 1) + \cdots + (n_1 + 2014)}{2013} = 2014$$

and solve for n_1. There were many other time-wasters in evidence here. In (b) some simplified the exponent without using the formula for triangular numbers by pairing factors 2^a and 2^{2015-a}, finding 1007 pairs—an elegant and efficient approach.

5. (a) Find all possible values of A so the polynomial

$$p(x) = (x^7 - 3x^5 + Ax^4 + 2x^3 + 2x + 1)(x^4 - 3x^3 - Ax^2 + x + 2)$$

satisfies $p(1) = -12$.

(b) Express the polynomial $f(x) = 2x^4 - x^3 + 2x^2 + x + 4$ in the form

$$A(x+1)^4 + B(x+1)^3 + C(x+1)^2 + D(x+1) + E.$$

Solution:

(a) Setting $x = 1$ we have the equivalent condition that $(1-3+A+2+2+1)(1-3-A+1+2) = (A+3)(1-A) = 3-2A-A^2 = -12$, or $A^2+2A-15 = (A+5)(A-3) = 0$. It follows that $A = 3$ are -5 are the only two solutions.

(b) One approach is to write $t = x + 1$, so that

$$\begin{aligned} f(x) = f(t-1) &= 2(t-1)^4 - (t-1)^3 + 2(t-1)^2 + (t-1) + 4 \\ &= 2(t^4 - 4t^3 + 6t^2 - 4t + 1) - (t^3 - 3t^2 + 3t - 1) + 2(t^2 - 2t + 1) + t + 3 \\ &= 2t^4 - 9t^3 + 17t^2 - 14t + 8 \\ &= 2(x+1)^4 - 9(x+1)^3 + 17(x+1)^2 - 14(x+1) + 8 \end{aligned}$$

Solution: (Alternative) Considering highest powers immediately gives $A = 2$. Now set $B(x+1)^3 + C(x+1)^2 + D(x+1) + E = f(x) - 2(x+1)^4 = -9x^3 - 14x^2 - 7x + 2$. Highest powers now give $B = -9$. Continue in this fashion.

Notes:

Another alternative approach uses Taylor's formula (a slight time advantage for students familiar with advanced topics in calculus, of which we saw a few). Very clever students having experience with synthetic substitution might even think of the following cascade of substitutions:

	2	−1	2	1	4
−1	2	−3	5	−4	8
−1	2	−5	10	−14	
−1	2	−7	17		
−1	2	−9			
−1	2				

Coefficients are read off the ends of the rows. I won't explain this mysterious array here; you may want to give it as a puzzle to students familiar with the method to see if they can perceive how it is constructed and unlock why it works. Those who successfully reverse engineer the method could be challenged to use the same approach to express the polynomial in powers of $x - 2$.

6. A single digit A and a single digit B are placed on the left and right, respectively, of the number 731 to form the five digit number $A731B$. If this five digit number is divisible by 36, find all possible pairs (A, B).

Solution: $A731B$ is divisible by 4 and 9. Therefore the last two digits form a multiple of 4 and the sum of all digits form a multiple of 9. The two digit number $1B$ is divisible by 4 for $B = 2$ and $B = 6$. If $B = 2$ the sum of all digits is $A + 13$, which is a multiple of 9 when $A = 5$. If $B = 6$, the sum of all digits is $A + 17$ which is a multiple of 9 when $A = 1$. Therefore the possible values for (A, B) are $(5, 2)$ and $(1, 6)$.

Notes:

An awkward long division could do this, or direct examination of $10^4A + 7 \cdot 10^3 + \cdots$. The sum of digits insight could be avoided by subtracting 7308 to yield $10^4A + B + 2$. 10^4 is divisible by 4 and leaves remainder 1 when divided by 9, so $B + 2$ is a multiple of 4 and $A + B + 2$ is divisible by 9. $B = 2$ gives $A = 5$ and $B = 6$ gives $A = 1$. Students found some inventive ways to waste time on brute-force analysis, writing out many possibilities and checking each with long division.

7. For the following system of equations:

$$\begin{array}{rcrcrcrcl} 2a & + & b & + & c & + & d & = & 5 \\ a & + & 2b & + & c & + & d & = & 7 \\ a & + & b & + & 2c & + & d & = & 2 \\ a & + & b & + & c & + & 2d & = & 6 \end{array}$$

(a) Solve for a, b, c and d.

(b) Show that there is no solution if every "2" appearing in the system is replaced by "-3"

Solution:

(a) Add all the equations to get $5a + 5b + 5c + 5d = 20$.
From this we get $a + b + c + d = 4$.
Subtract $a + b + c + d = 4$ from the original equations to get

$$a = 1, b = 3, c = -2, d = 2$$

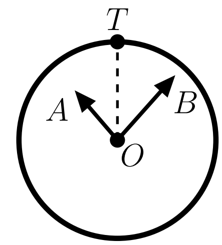
(b) Adding the equations gives $0 = 15$ (or $0 = 20$ if the instruction is understood to apply only to the "2"s on the left hand side)—a contradiction, so there is no solution.

Solution: (Alternative) For (a), subtracting in pairs gives $b - a = 2$, $b - c = 5$, $d - c = 4$ or $(b, c, d) = (a + 2, a - 3, a + 1)$. Plugging into the first equation gives $5a + 2 - 3 + 1 = 5$, or $a = 1$ and the rest follows.

Notes:

Trying Gaussian elimination may mess things up more than help. In part (b) some students were unsure whether the "2" on the right side of the third equation should be replaced by "-3". We had anticipated this possibility and deliberately built the problem so that it made no essential difference which interpretation was used, and no marks were lost by students who took this approach.

8. Let O be the centre of a standard clock face and let T be the point at the 12 o'clock position. Let A be the end point of the hour hand and B the end point of the minute hand, as shown. At what time between 10:00 o'clock and 11:00 o'clock, accurate to the nearest second, is $\angle AOT$ equal to $\angle BOT$?



Solution: At 10 o'clock $\angle AOT = 60°$ and $\angle BOT = 0°$. $\angle AOT$ decreases at a rate of $30°$ per hour and $\angle BOT$ increases at a rate of $360°$ per hour. Therefore after time t (in hours) has elapsed $\angle AOT = 60 - 30t$ and $\angle BOT = 360t$. Equating these two expressions and solving for t gives $t = \frac{2}{13}$ hours $= 9\frac{3}{13}$ minutes $= 9$ minutes and $13\frac{11}{13}$ sec. To the nearest second, the time is $10:09:14$.

Notes:

Students could (and did) successfully solve this with a reasonable amount of work by picking times such as 10 : 09 and 10 : 10 using some coarse analysis, and working out angles. By narrowing the gap in sensible ways the answer can be determined within 1 second in only a few steps. Students converting to radians were wasting their time. Some students converted fractions to decimals, rounding to a couple of places—and generally lost a mark for round-off inaccuracy, giving the wrong number of seconds.

Some students solved for a different time at which the two angles were equal, which came to 10:54:33. This answer (with work shown) was only worth 5 marks, however, because of the wording of the instruction: "as shown". (Note: It is not generally the case that mathematical meaning should not depend on how a diagram is drawn, but in this case we are concerned about students taking seriously the specifics of a problem as posed. One must not always expect problems to adhere to strict conventions of formal mathematics.

However, full marks were given to students solving in both ways, particularly if notes were made to this effect. In one case the inclusion of both cases was a consideration in breaking a tie, though no bonus marks were awarded for this although the markers were impressed with a couple of very nice writeups.

9. Prove that $\sqrt{4+2\sqrt{3}}+\sqrt{28-10\sqrt{3}}$ is an integer.

Solution: $4+2\sqrt{3}=1+2\sqrt{3}+3=\left(1+\sqrt{3}\right)^2$ and $28-10\sqrt{3}=25-10\sqrt{3}+3=\left(5-\sqrt{3}\right)^2$.
Therefore $\sqrt{4+2\sqrt{3}}+\sqrt{28-10\sqrt{3}}=(1+\sqrt{3})+(5-\sqrt{3})=6$, which is an integer.

Notes:

A lucky guess that both terms are of the form $a+b\sqrt{3}$, integers a, b yields solutions with a bit of work. A bit of insight shows that the two "b" values have to add to zero, so the second one drops out even faster. We did not think that many students would perceive the squares under the radicals very easily, but were pleased to see that a number of students seemed to have this insight.

Some students will take is to square the whole expression in the hopes that some cancellation will happen, and the result will be the square of an integer. Unfortunately, this leads to much work, many ways to go astray, and if not carefully managed, a series of relations that do not simplify nicely. Students trying to "rationalize the numerator" only further complexified the task, and were unable to complete it.

10. Consider the equation $7a + 12b = c$ where a, b and c are nonnegative integers. For many values of c it is possible to find one or more pairs (a, b) satisfying the equation. Given $c = 26$, for example, $(a, b) = (2, 1)$ is the only solution.

(a) If $c = 365$, find all possible solutions (a, b), where a and b are nonnegative integers.

(b) There are some values of c for which no solutions exist. For example, there is no pair of nonnegative integers (a, b) such that $7a + 12b = 20$, so $c = 20$ is one such case. Find the largest value of c for which there are no solutions.

Solution:

(a) We require lattice points on the line $7x + 12y = 365$. Solving for y gives us $y = \frac{365-7x}{12} = 30 + \frac{5-7x}{12}$. In order for y to be an integer $5 - 7x = 12 - 7(x + 1)$ must be a multiple of 12. Equivalently, $7(x + 1)$ must be a multiple of 12, so $x = 12k - 1$ for some $k \in Z$—and thus $y = 31 - 7k$. Since $x, y \geq 0$ we have $\frac{1}{12} \leq k \leq \frac{31}{7}$ so $k \in \{1, 2, 3, 4\}$ and the complete set of solutions is $(a, b) \in \{(11, 24), (23, 17), (35, 10), (47, 3)\}$.

(b) Note that $7(-7) + 12(4) = -1$ and that $7(5) + 12(-3) = -1$.

Start with an arbitrarily large value of c for which $7a + 12b = c$ does have at least one solution. Then $7(a - 7) + 12(b + 4) = c - 1$ And $7(a + 5) + 12(b - 3) = c - 1$. So if we have a solution for c, and if $a > 7$, then we can find a solution for $c - 1$ by decreasing a by 7 and increasing b by 4. If $a < 7$ but $b > 3$, we can still find a solution for $c - 1$ by increasing a by 5 and decreasing b by 3. Thus we can generate solutions for $c - 1, c - 2, c - 3$, and so on until we reach the point where the above algorithm fails because $a < 7$ and $b < 3$. This happens when we have reached the number $7(6) + 12(2) = 66$. Thus, we cannot use our algorithm for finding solutions for $c = 65$. It is easily confirmed that $7a + 12b = 65$ has no solutions by trying the only possible values for b, i.e. $1, 2, 3, 4$ and 5. So 65 is the largest value of c for which $7a + 12b = c$ has no solutions.

Solution: (Alternative (b)) If $7a + 12b = c$, then $7(a + 1) + 12b = c + 7$. Thus, solutions for each of $c = 66, 67, 68, 69, 70, 71, 72$ (easily produced) imply solutions for all $c > 65$. Now, if we demonstrate that there are no solutions for $c = 65$, we are done.

Notes:

Various tricks can be used to simplify individual steps by looking at modular residues. We saw a number of creative solutions.