1. (a) Find three consecutive even integers with a sum of 36.
   (b) Find three consecutive integers so that smallest plus twice the largest is 15 more than the middle integer.

Solution: (a) Their average, $\frac{36}{3} = 12$, must be the middle number, so the numbers are 10, 12 and 14.
(b) Let n be the middle number. Then $(n-1) + 2(n+1) = 3n + 1 = n + 15$, so $n = 7$, and the numbers are 6, 7 and 8.

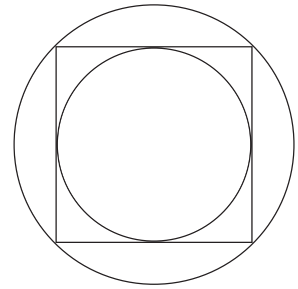
2. (a) Find the number half way between $\frac{2}{5}$ and $\frac{5}{8}$.

Solution: $\frac{1}{2}\left(\frac{2}{5}+\frac{5}{8}\right)=\frac{2\cdot 8+5\cdot 5}{2\cdot 5\cdot 8}=\frac{41}{80}$

(b) What single percentage increase is equivalent to a series of two increases of 10%?

Solution: $(1+0.1)^2-1=1.21-1=0.21=21\%$.

3. (a) A circle has an area of 5 square units. A square is constructed around the circle so that each side of the square is tangent to the circle. A larger circle is constructed around the square so that it passes through all four vertices of the square. Find the area of the larger circle.



Solution: Let each side of the square have length $2x$. Then the small circle has area πx^2 and the large circle has area $\pi(\sqrt{2}x)^2$. Thus the larger circle has twice the area of the smaller one, or 10 square units.

(b) A rectangular tank has length 2 metres, width 3 metres, and height 3 metres and is full of water. A cylindrical tank has radius 2 metres and height 3 metres, and is empty. Water is siphoned from the rectangular tank to the cylindrical tank. The siphon will stop flowing when the depth of water is the same in both tanks. What will be the depth at that moment?

Solution: The total volume is $(2)(3)(3) = 18$ m^3. If the common depth is h, then $2 \cdot 3 \cdot h + \pi(2^2)h = 18$. Solve the equation to get $h = \frac{9}{3+2\pi}$.

4. (a) The digits 1, 2, 3, 4 can be arranged to form a four digit number in 24 different ways. What is the sum of these 24 four digit numbers?

Solution: (Method #1) If we use the standard addition algorithm, each column from the units to the thousands consists of six 1s, six 2s, six 3s and six 4s, with a total of $6(1+2+3+4) = 60$. Thus when we add the units we write down the 0 and carry the 6. When we add the tens (along with the carry from the units) we get 66, so we write down 6 and carry 6. Continuing in this way, the sum is 66660.

(Method #2) The 24 numbers can be divided into 12 pairs by the following rule: Pair each number with the number formed by replacing 1 with 4, 2 with 3, 3 with 2, and 4 with 1. For example 1432 would be paired with 4123. Each pair has a sum of 5555, so the total of all 24 numbers is $5555(12) = 66660$.

(b) Write $x^4 + 4$ as the product of two quadratic polynomials with integer coefficients.

Solution: $x^4 + 4 = x^4 + 4x^2 + 4 - 4x^2 = (x^2 + 2)^2 - (2x)^2$. Factoring the difference of squares gives us $(x^2 + 2x + 2)(x^2 - 2x + 2)$.

5. (a) Find the value of x for which $\sqrt{x} - 1$ and $\sqrt{x} + 1$ are reciprocals.

Solution: $(\sqrt{x} - 1)(\sqrt{x} + 1) = 1$ so $x = 1 = 1$; $x = 2$.

(b) Prove that there is no real value of x for which x and $1 - 2x$ are reciprocals.

Solution: Assume $x(1 - 2x) = 1$. Then $x - 2x^2 = 1$, so $2x^2 - x + 1 = 0$. The discriminant of this quadratic equation is $(-1)2 - 4(2)(1) = -7$. A negative discrimant shows that there are no real roots.

6. To represent a number in a base system other than 10, say base $B > 0$, let us write "S_B" where S is a string of suitable digits in base B (that is, elements of the set $\{0, 1, 2, \ldots, B-1\}$). (If no base is given it is understood that a number is written in base 10.) Thus, $112_3 = 1 \cdot 3^2 + 1 \cdot 3 + 2 = 14$ and $1036_7 = 1 \cdot 7^3 + 0 \cdot 7^2 + 3 \cdot 7 + 6 = 370$.

   (a) In what base B is 213_B equal to 58?

   (b) Evaluate the following expression. Report your answer in base 10.

$$2012_{\left(22_{\left(12_{\left(21_{(11_2)}\right)}\right)}\right)}$$

Solution: (a) Let $2B^2 + B + 3 = 58$. Either by trial and error or by solving the quadratic $2B^2 + B - 55 = (B-5)(2B+11) = 0$, we obtain $B = 5$.

(b)

$$\begin{aligned} 2012_{\left(22_{\left(12_{\left(21_{(11_2)}\right)}\right)}\right)} &= 2012_{\left(22_{\left(12_{(21_3)}\right)}\right)} \\ &= 2012_{\left(22_{(12_7)}\right)} = 2012_{(22_9)} = 2012_{20} \\ &= 2 \cdot 20^3 + 20 + 2 = 8022. \end{aligned}$$

7. In the polynomial equation

$$x^7 + 3x^6 + 2x^5 + 4x^4 + 12x^3 + x^2 + x - 2 = (x^2 + 3x + 2)q(x) + ax + b,$$

find the values of a and b.

Solution: By long division (which only requires 3 steps for this problem) $q(x) = x^5 + 4x^2 - 7$ and $ax + b = 22x + 12$. So $(a, b) = (22, 12)$.

8. A line with positive slope passes through the origin and is tangent to the circle $(x-5)^2+y^2=9$. Find the equation of this line.

Solution: (Method #1) (Of the many solutions, this one is likely the easiest, but not the one students are most likely to use.) Let O be the origin, $A(5,0)$ the centre of the circle and C the point of tangency. $OA=5$, $OC=3$, and, since the radius is perpendicular to the tangent, we can use Pythagoras to get $OC=4$. The slope of the line is equal to tangent of the angle of inclination. Thus the line has a slope of $\frac{3}{4}$. It has equation $y=\frac{3}{4}x$.
(Method #2) (Sketched) Start with the system of equations $y=mx$ and $(x-5)^2+y^2=9$. Substitute mx for y into the second equation and get a quadratic in x. Since tangency occurs, this quadratic has only one solution. We can therefore let the discriminant equal 0, and solve for m.
(Method #3) The perpendicular distance from $(5,0)$ to $y=mx$ is 3. Use the standard equation for distance from point to line, and solve for m.

9. Let A be the point $(4, 9)$ and B be the point $(10, 5)$. A very small insect starts at A, crawls to a point C on the x-axis, and then crawls to the point B. If he took the shortest possible path to do this, find the distance travelled and the co-ordinates of the point C.

Solution: $D(10, -5)$ is the reflection of B in the x-axis. The shortest distance from A to D is a straight line of length $\sqrt{(10-4)^2 + (-5-9)^2} = \sqrt{232} = 2\sqrt{58}$. Since CB is the same distance as CD, $2\sqrt{58}$ will be the minimum value for $AC + CB$. The co-ordinates of C will be the x-intercept of the line AD. AD has slope $\frac{9+5}{4-10} = -\frac{7}{3}$ and equation $y - 9 = -\frac{7}{3}(x-4)$. Letting y = 0 we solve to get $x = \frac{55}{7}$. The coordinates of C are $(\frac{55}{7}, 0)$.

10. A square of area 1 is divided into three rectangles which are geometrically similar (i.e., they have the same ratio of long to short sides) but no two of which are congruent. Write A, B and C for the areas of the rectangles, ordered from largest to smallest. Prove that $(AC)^2 = B^5$.

Solution: Only one such division is possible: One rectangle must use two of the corners of the square and therefore has size $1 \times a$ for some number $a < 1$. The other two are $b \times (1-a)$ and $c \times (1-a)$ where $b + c = 1$ (with $b > c$). The side ratios give

$$\frac{a}{1} = \frac{1-a}{b} = \frac{c}{1-a}.$$

Cross multiplying and eliminating c gives the following series of calculations:

$$\begin{aligned}
ab &= 1-a \\
c &= 1-b = a(1-a) \\
b &= a^2 + 1 - a = a^2 + ab \\
a^2 &= b(1-a) = ab^2 \\
a &= b^2 \\
AC &= a(1)(1-a)(1-b) = a^2(1-a)^2 = a^2(ab)^2 = a^4b^2 = a^5
\end{aligned}$$

Further, $B = b(1-a) = ab^2 = a^2$. It follows that $(AC)^2 = a^{10} = B^5$.