2011 Manitoba Mathematical Competition
SOLUTIONS

1. (a) A square has a diagonal 10 cm long. Find the area of the square.

Solution: Direct calculation or cut on both diagonals and reassemble into two squares of side 5. Total area: 50.

(b) $ABCD$ is a square. AB and CD are increased by 10% while BC and DA are decreased by 10%. By what per cent does the area change?

Solution: The area, $\mathcal{A}$, of $ABCD$ becomes $\mathcal{A} \cdot (1.10 \cdot 0.9 - 1) = -0.01\mathcal{A}$. So it changes by -1%.

2. On a number line the point A is at $\frac{1}{5}$ and the point B is at $\frac{1}{3}$.

(a) The point C is half way between A and B. Where is C located?

Solution: Halfway between A and B is at the mean of their locations: $\frac{1}{2}\left(\frac{1}{5}+\frac{1}{3}\right) = \frac{1(3+5)}{2 \cdot 3 \cdot 5} = \frac{4}{15}$.

(b) Point D is to the right of B. If $BD = 2(AB)$, find the location of D.

Solution: Let D be at x. Then $x - \frac{1}{3} = 2\left(\frac{1}{3} - \frac{1}{5}\right)$, so D is at $x = \frac{1}{3} + \frac{4}{15} = \frac{3}{5}$.

3. (a) Solve for x: $\frac{1}{4}\left(2^{4x}\right) = 32$

Solution: $2^{4x-2} = 32 = 2^5$, $4x - 2 = 5$, $x = \frac{7}{4}$.

(b) Solve for x and y:

$$\begin{aligned} 2^x - y &= 7 \\ 2^{x+1} + 2y &= 2 \end{aligned}$$

Solution:

$$\begin{aligned} 2(2^x - y &= 7) \\ +(2^{x+1} + 2y &= 2) \\ \hline 2 \cdot 2^{x+1} &= 16 \\ 2^{x+2} &= 2^4 \\ x &= 2 \\ 2^2 - y &= 7 \\ y &= -3 \end{aligned}$$

So $(x, y) = (2, -3)$.

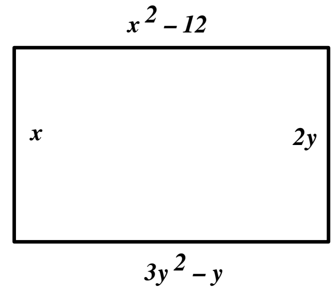
4. (a) Let n be the product of the first 30 positive integers. In how many zeros does the base ten representation of n end?

Solution: The multiples of 5 contributing to this product are 5, 10, 15, 20, 25 and 30, so the highest power of 5 that divides n is $5 \cdot 5 \cdot 5 \cdot 5 \cdot 5^2 \cdot 5 = 5^7$. Since n is also divisible by $8 \cdot 16 = 2^7$ we see that $10^7 | n$ whereas 10^8 does not. So n ends in seven zeros.

(b) Find a pair of positive integers (a, b) that satisfies $a^2 - b^2 = 2011$.

Solution: $(a+b)(a-b) = 2011$. Setting $a+b = 2011$ and $a-b = 1$ and adding we obtain $2a = 2012$, or $a = 1006$. Then $b = 1005$, which is directly verified to be a solution.

5. (a) The sides of a rectangle are as shown. Find a numerical value for the area of the rectangle.



Solution: Equating opposite sides,

$$x = 2y$$
$$x^2 - 12 = 3y^2 - y$$

Substituting, $(2y)^2 - 12 = 3y^2 - y$, so $y^2 + y - 12 = 0$. $(y+4)(y-3) = 0$. Since $y > 0$, $y = 3$ and the required area is $(3y^2 - y)(2y) = (6^2 - 12) \cdot 6 = 144$ square units

(b) Prove that, for every positive integer n, the number $n^3 + 2n$ is divisible by 3.

Solution: By cases, starting with n having remainder $0, 1$ or 2 upon division by 3. For example, taking $n = 3k+1$ we have $n^3 + 2n = (3k+1)^3 + 2(3k+1) = 27k^3 + 27k^2 + 9k + 1 + 6k + 2 = 3(9k^3 + 9k^2 + 5k + 1)$, a multiple of 3 as required.

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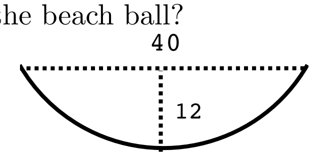
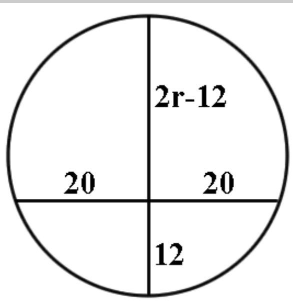
Alternatively, rewrite as $n(n-1)(n+1) + 3n$. The second factor is obviously a multiple of three and the first is a product of three consecutive numbers, one of which must therefore be a multiple of 3. The result follows.

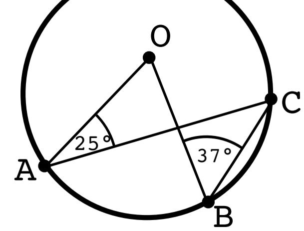
6. (a) A beach ball, floating on a lake, was not removed until the water had frozen. It left an impression in the ice that was 12 cm deep and 40 cm across. What was the radius of the beach ball?



Solution: (By PIP) $12(2r - 12) = 20^2$, giving $r = \frac{68}{3}$. The ball's radius was $22\frac{2}{3}$ cm.



(b) O is the centre of the circle, $\angle OAC = 25°$ and $\angle OBC = 37°$. Find $\angle AOB$.



Solution: Let $\angle C = x^\circ$. Then $\angle O = 2x^\circ$ (stands on common chord AB). Now,

$$\begin{aligned}\angle A + \angle O &= \angle B + \angle C\\ 25 + 2x &= 37 + x\\ x &= 12\end{aligned}$$

So $\angle AOB = 2x = 24^\circ$.

7. $56a = 65b$, where a, b are positive integers. Prove that $a + b$ is composite.

Solution: Add $56b$ to both sides, giving $56(a + b) = 11^2 b$. Since 11 and 56 are relatively prime we must have $11^2 | (a + b)$, so $(a + b)$ is composite.

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Alternatively: Since 56 and 65 are relatively prime, 56 divides b and 65 divides a. Let $b = 56B$ and let $a = 65A$. Substituting back in we get: $56(65A) = 65(56B)$, so $A = B$. Therefore $a + b = 56A + 65A = 121A$, clearly composite.

8. Find the least positive integer n for which $\dfrac{n - 11}{3n + 8}$ is a nonzero reducible fraction (i.e., not in lowest terms).

Solution: Let $d > 1$ be a common divisor of $n-11$ and $3n+8$. Then d is a divisor of $3n+8-3(n-11) = 41$. Since 41 is prime and $d > 1$ it must be that $n = 41a + 11$. The least such value is $n = 52$.

9. A function f satisfies the equation $f(x + 1) = \frac{2f(x)+x}{3} + 1$ for all real numbers x. Suppose $f(1000) = 2011$. Find the value of $f(2011)$.

Solution: Suppose for some n, $f(n) = n + a$. Then $f(n + 1) = \frac{2(n+a)+n}{3} + 1 = (n + 1) + \frac{2}{3}a$. It follows that, for any $k \in \mathbb{Z}^+$, $f(n + k) = n + k + \left(\frac{2}{3}\right)^k a$. Since $f(1000) = 1000 + 1011$, we can take $n = 1000$, $a = 1011$. Taking $k = 1011$ gives

$$f(2011) = f(1000 + 1011) = 1000 + 1011 + \left(\frac{2}{3}\right)^{1011} \cdot 1011 = 2011 + 1011 \left(\frac{2}{3}\right)^{1011}$$

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Alternative arrangement of this solution: $f(x) = x$ satisfies the first equation. Let $g(x) = f(x) - x$. Then $g(x + 1) = \cdots = \frac{2}{3}g(x)$, so $g(x + k) = (2/3)^k g(x)$ for $k = 1, 2, 3, \ldots$. Put $x = 1000$, $k = 1011$.

10. There exist positive integers whose value is quadrupled by moving the rightmost decimal digit into the leftmost position. Find the smallest such number.

Solution: Write the number as $N = 10x + a$, where $a \in \{1, 2, \ldots, 9\}$ (if $a = 0$ the described property obviously fails). Let n be the number of digits in x. The described property of N may be expressed $10^n a + x = 4(10x + a)$. Thus, $x = \frac{a(10^n-4)}{39}$. Now, $39 = 3 \cdot 13$. While it may be that $3|a$, of necessity a and 13 are relatively prime, so we must have $13|(10^n - 4)$. This first happens when $n = 5$: $10^5 - 4 = 99996 = 13 \cdot 7692$. Thus $x = \frac{7692a}{3} = 2564a$ From $n = 5$ we may infer that $a \in \{4, 5, 6, 7\}$. The smallest of these values works, for $x = 2564 \cdot 4 = 10256$ and $N = 102564$ has the required property, $4N = 410256$, so this is the smallest such number.