2010 Manitoba Mathematical Competition
SOLUTIONS

1. (a) If $x^2 - y^2 = 39$ and $x + y = 3$, find $x - y$.

Solution: $(x-y)(x+y) = 29 = 3(x-y) = 39$, so $x - y = 13$.

(b) Solve for x: $x + \sqrt{x} = 20$

Solution: $(\sqrt{x}+5)(\sqrt{x}-4) = 0$, $\sqrt{x} = 4$, so $x = 16$.

2. (a) A performer asks the members of his audience to think of a number. They are to increase this number by 3. The result is to be multiplied by 2. 10 is subtracted from the new result. This latest result is divided by 2. When told the final result an audience member obtains, he immediately tells them their original number. What one-step formula can he use to convert the final result back into the original number?

Solution: Let x be the original number. In order, the results are $x + 3$, $2x + 6$, $2x - 4$, $x - 2$. Adding 2 to the final result, $x - 2$, will bring us back to the original number, x.

(b) A fair coin is tossed four times. What is the probability that it shows heads three times and tails only once?

Solution: There are $2^4 = 16$ possible outcomes, four of which show exactly three heads, so the required probability is $\frac{4}{16} = \frac{1}{4}$.

3. (a) At what points does the circle with equation $x^2 + y^2 = 3$ intersect the parabola with equation $y = 2x^2$?

Solution: Eliminating y gives $x^2 + (2x^2)^2 = 3$, or $4x^4 + x^2 - 3 = (x^2+1)(4x^2-3) = 0$, so $x^2 = -1$ or $\frac{3}{4}$, the first of which is ruled out for real x. So $x = \pm\frac{\sqrt{3}}{2}$, $y = \frac{3}{2}$, and the points are $\left(\pm\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$.

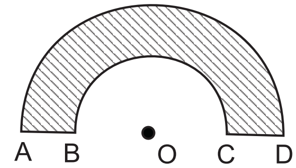
Alternate #1: Multiply the first equation by 2 and eliminate x: $2y^2 + y - 6 = (2y-3)(y+2) = 0$, so $y = \frac{3}{2}$ or -2. Now $y = 2x^2$ gives $(x, y) = \left(\pm\frac{\sqrt{3}}{2}, \frac{3}{2}\right)$ in the first case, and eliminates the second.

(b) The three lines whose equations are $y = x - 7$, $x + y = 3$ and $y = kx + 8$ pass through a common point. Find the value of k.

Solution: The first two meet at $(5, -2)$ (add to eliminate x, or subtract or substitute to eliminate y). Substitute in the third: $-2 = k(5) + 8$. So $k = -2$.

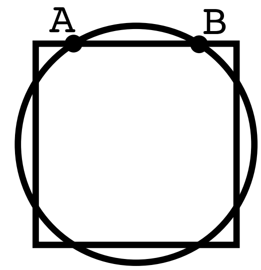
Alternate #1: (Gaussian Elimination) $\left(\begin{array}{cc|c} 1 & -1 & 7 \\ 1 & 1 & 3 \\ k & -1 & 8 \end{array}\right) \equiv \left(\begin{array}{cc|c} 1 & 0 & 5 \\ 0 & 1 & -2 \\ 0 & 0 & -10-5k \end{array}\right)$, so $k = -2$.

4. (a) Two semicircles, as shown, have centre O. Points A, B, O, C and D are colinear with $AB = CD = 1$. The shaded area is 15π. Find the length of AD.



Solution: Let $r = AO$. $\pi r^2 - \pi(r-1)^2 = 30\pi$, so $2r - 1 = 30$, so $r = \frac{31}{2}$, and $AD = 2r = 31$.

(b) Consider a circle and a square whose areas are equal and which have the same centre of symmetry (see diagram). If the radius of the circle is 2, find the length of AB. (Express your answer in terms of π.)



Solution: If x is the side of the square, we obtain $\pi(2^2) = x^2$, so $x = 2\sqrt{\pi}$. A vertical from the centre O to the midpoint C of AB forms right triangle C with hypotenuse $OB = 2$, side $OC = \sqrt{\pi}$ and other side $CB = \sqrt{4 - \pi}$. So $AB = 2CB = 2\sqrt{4 - \pi}$.

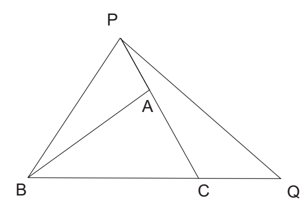
5. (a) The number 64 is both a perfect square and a perfect cube. Find the next positive integer with this property.

Solution: Since the number is both a perfect square and a perfect cube it must be a perfect 6th power; conversely, any 6th power has this property. The first case after $2^6 = 64$, then, is $3^6 = 729$

(b) How many solutions of $2x + 3y = 763$ are there in positive integers x, y?

Solution: Count the odd values of y such that $3y$ does not exceed 763: $\lfloor \frac{763}{3} \rfloor = 254$ There are exactly 127 odd numbers $y \leq 254$, so there are 127 solutions of the required type.

6. In the diagram, $AP = \frac{1}{3}PC$ and $CQ = \frac{1}{2}BC$. Prove that the area of $\triangle BPA$ is two-thirds times the area of $\triangle CPQ$.



Solution: Take PA as a base for $\triangle BPA$ and PC as a base for $\triangle CPQ$. Dropping altitudes BB' to PA from B and QQ' to PC from Q, we see that triangles $\triangle BB'C$ and $\triangle QQ'C$ are similar because of opposite angles at C and because $B'Q'$ crosses parallel lines BB' and QQ'. It follows that $BB' = 2QQ'$. So the area of $\triangle BPA$ is $\frac{1}{2}(PA)(BB') = \frac{1}{2}(\frac{1}{3}PC)(2QQ') = (\frac{2}{3})(\frac{1}{2}(PC)(QQ'))$, or two-thirds the area of $\triangle CPQ$, as required.

Alternate #1: $AP = \frac{1}{3}PC$, which implies $|\triangle BAP| = \frac{1}{3}|\triangle BPC|$. $CQ = \frac{1}{2}BC$, which implies $|\triangle CPQ| = \frac{1}{2}|\triangle BPC|$. Combining these we obtain $|\triangle BAP| = \frac{2}{3}|\triangle CPQ|$.

7. Five distinct integers are added in pairs, giving the ten sums

$$7, 11, 12, 13, 14, 18, 21, 22, 26, 28.$$

Find the numbers, justifying your answer with a series of deductions clearly demonstrating that there is no other possibility.

Solution: Let the original numbers be ordered as follows: $a < b < c < d < e$. Each number can be paired with 4 others. Therefore $4a+4b+4c+4d+4e = 7+11+\cdots+28 = 172$, so $a+b+c+d+e = 43$. The largest and smallest sums are $d+e = 28$ and $a+b = 7$ so $7+c+28 = 43$, and $c = 8$. The second largest sum is $c+e = 26$ so $e = 26-8 = 18$. Thus $d = 28-e = 28-18 = 10$. Similarly the second smallest sum is $a+c = 11$, so $a = 11-c = 3$, and $b = 7-a = 7-3 = 4$. The numbers are $(a, b, c, d, e) = (3, 4, 8, 10, 18)$.

Alternate #1: [first obtain as before] $a = 11-c$, $b = c-4$, $d = c+2$, $e = 26-c$. [Then one of:]

(a) $a+d = 13$, which is fourth on the list of sums. The third sum 12 is too small to be $x+d$ or $x+e$, so the only way to get this sum is $b+c = 12$.

(b) The only odd sums are $7 = a+b$, $11 = a+c$, $13 = a+d$ and 21. But $a+e = 37-2c$ is odd, so this must be 21.

(c) $d = c+2$. The only sums that differ by 2 are $(11, 13)$ and $(12, 14)$ and $(26, 28)$, which must be $(x+c, x+d)$ for $x = a, b, e$. So $b+c = 12$.

Alternate #2: Let m of the original numbers be odd, n even. Then $m+n = 5$. A sum of two integers is odd if and only if one summand is even the other odd, so $mn = 4$, the number of odd sums. Solving gives $(m, n) = (1, 4)$ or $(4, 1)$. Let $a < b < c < d$ be the four numbers of common parity, and x be the remaining one. The odd sums give $(a, b, c, d) = (7-x, 11-x, 13-x, 21-x)$. Further, ten sums add to $4(x+a+b+c+d) = 172$, so $x+(7-x)+(11-x)+(13-x)+(21-x) = 52-3x = 43$. So $x = 3$ and the original numbers are $4, 8, 9, 10, 18$.

8. A line with slope 1 meets the parabola $y = x^2$ at A and B. If the length of segment AB is 3 what is the equation of that line?

Solution: Let the points of intersection be (a, a^2), (b, b^2), $a < b$. The slope of the line is therefore $\frac{b^2-a^2}{b-a} = a+b = 1$. The square of the distance between the points is

$$(b-a)^2 + (b^2-a^2)^2 = (b-a)^2 + ((a+b)(b-a))^2 = (b-a)^2 + 1 \cdot (b-a)^2 = 2(b-a)^2 = 9,$$

so $b-a = \pm\frac{3}{\sqrt{2}}$. Since $a < b$, $b-a = \frac{3}{\sqrt{2}}$. Adding $a+b = 1$ we obtain $b = \frac{1}{2} + \frac{3}{2\sqrt{2}}$. So $b^2 = \frac{11}{8} + \frac{3}{2\sqrt{2}}$. In point-slope form the line is $y - b^2 = (1)(x-b)$, or $y - \left(\frac{11}{8} + \frac{3}{2\sqrt{2}}\right) = x - \left(\frac{1}{2} + \frac{3}{2\sqrt{2}}\right)$, which simplifies to $y = x + \frac{7}{8}$.

Alternate #1: Let the line be $y = x+k$. The points of intersection, A and B, will have x-coordinates which are solutions to $x^2 = x+k$. Rewrite this as $x^2 - x - k = 0$. From the quadratic formula we see that the two roots of ax^2+bx+c differ by $\dfrac{\sqrt{b^2-4ac}}{a}$. Applying this result to our quadratic, we see that the x-coordinates of A and B differ by $\sqrt{1+4k}$. Since AB has slope 1, the y-coordinates differ by the same amount. Since $AB = 3$, we can apply Pythagoras to get $\left(\sqrt{1+4k}\right)^2 + \left(\sqrt{1+4k}\right)^2 = 2(1+4k) = 9$. Solve to get $k = \frac{7}{8}$, so the line is $y = x + \frac{7}{8}$.

9. Solve for x and y:

$$x + y + xy + 2 = 0$$
$$x^2 + y^2 + x^2y^2 - 16 = 0$$

Solution: $x + y = -(xy + 2)$, so $x^2 + 2xy + y^2 = x^2y^2 + 4xy + 4$. Use the second equation to eliminate $x^2 + y^2$:

$$(16 - x^2y^2) + 2xy = x^2y^2 + 4xy + 4$$
$$2x^2y^2 + 2xy - 12 = 0$$
$$x^2y^2 + xy - 6 = 0$$
$$(xy + 3)(xy - 2) = 0$$

So $xy = -3$ or $xy = 2$.
If $xy = -3$ the first equation gives $0 = x + y - 1 = x - \frac{3}{x} - 1$, so $x^2 - x - 3 = 0$, so $x = \frac{1 \pm \sqrt{13}}{2}$, and $y = 1 - x = \frac{1 \mp \sqrt{13}}{2}$. If $xy = 2$, We similarly obtain $x + y + 4 = 0 = x + \frac{2}{x} + 4 = 0$, so $x^2 + 4x + 2 = 0$, $x = -2 \pm \sqrt{2}$ while $y = -4 - x = -2 \mp \sqrt{2}$.
There are, therefore, four solutions: $\left(\frac{1 \pm \sqrt{13}}{2}, \frac{1 \mp \sqrt{13}}{2}\right)$, $(-2 \pm \sqrt{2}, -2 \mp \sqrt{2})$

Alternate #1: Let $s = x + y$, $p = xy$. Then $x^2 + y^2 = s^2 - 2p$, and the equations become $s + p + 2 = 0$, $s^2 - 2p + p^2 - 16 = 0$. Eliminate p to obtain $2s^2 + 6s + 9 = 17$; $x^2 + 3s - 4 = (s + 4)(s - 1) = 0$, so $(s, p) = (1, -3)$ or $(-4, 2)$. Then x and y are roots of $t^2 - t - 3 = 0$ or $t^2 + 4t - 2 = 0$, yielding the same solutions.

Alternate #2: The first relation can be written $y(1 + x) = -x - 2$. Squaring gives $y^2(1 + x)^2 = x^2 + 4x + 4$. Multiplying the second relation by $(1 + x)^2$ and substituting gives

$$x^2(x^2 + 2x + 1) + (x^2 + 4x + 4) + x^2(x^2 + 4x + 4) = 16(x^2 + 2x + 1),$$

which simplifies to $x^4 + 3x^3 - 5x^2 - 14x - 6 = (x^2 - x - 3)(x^2 + 4x - 2) = 0$. Immediately we obtain the four values of x, and the corresponding values of y are obtained by substitution.

10. All three sides of a right triangle are integers. Prove that the area of the triangle:

(a) is also an integer;
(b) is divisible by 3;
(c) is even.

Solution: Let the three sides be a, b and c, with $a^2 + b^2 = c^2$. The area is $\frac{1}{2}ab$. (a) This will be an integer if one of a, b is even. Suppose a, b are both odd. Then $a = 2h + 1$, $b = 2k + 1$, where $h, k \in \mathbb{Z}$. Thus $a^2 + b^2 = (2h + 1)^2 + (2k + 1)^2 = 4(h^2 + k^2 + h + k) + 2 = c^2$. But this is impossible, since the square of any integer leaves a remainder of either 0 or 1 when divided by 4.
(b) Each integer is of one of the three forms $3k, 3k + 1$ and $3k + 2$, so the square of an integer is one of the forms $9k^2, 9k^2 + 6k + 1$, and $9k^2 + 12k + 3 + 1$. Thus, if neither a nor b is divisible by 3, then a^2 and b^2 both leave remainder 1 when divided by 3; so $a^2 + b^2$ leaves a remainder of 2. But $a^2 + b^2 = c^2$, which can only leave a remainder of 0 or 1. So at least one of a or b is divisible by 3. By part (a) one of them is even. It follows that $A = \frac{1}{2}ab$ is a multiple of 3.
(c) If a and b are both even then we are done. So let (WLOG) $a = 2h+1$. From (a), b cannot also be odd, so $a^2+b^2 = c^2$ is odd, so c is odd, say $c = 2k+1$. Then $b^2 = c^2-a^2 = 4(k-h)(k+h+1)$. Now, $k + h + 1$ and $k - h$ differ by $2h + 1$, so one of them is even and the other odd. Therefore, 8 divides b^2. So b is a multiple of 4, and the result follows.