Manitoba Mathematical Competition
SOLUTIONS

1. (a) The average of x and y is 3. What is the value of z if $3x - z = z - 3y$?

Solution:

$$2z = 3x + 3y,$$

so

$$z = 3\frac{x+y}{2} = 3 \cdot 3 = 9.$$

(b) If real numbers a and b satisfy both $a - b = 3$ and $ab = 2$, what is the value of $\frac{1}{a} - \frac{1}{b}$?

Solution:

$$\frac{1}{a} - \frac{1}{b} = \frac{b-a}{ab} = \frac{-(a-b)}{ab} = -\frac{3}{2}.$$

Solution: (Note) It is possible, but unnecessary (and unnecessarily complicating) to find the values of a and b.

2. (a) Find the area bounded by the x-axis, the y-axis and the line $5x + 4y = 20$.

Solution: The intercepts are $x = 4$, $y = 5$, so this is a right triangle of side lengths 4 and 5; its area is

$$A = \frac{1}{2} \cdot 4 \cdot 5 = 10.$$

(b) A circle has diameter AB, where A is the point $(3, 5)$ and B is the point $(5, 9)$. A line through the origin divides this circle into two regions of equal area. Find the slope of that line.

Solution: The line must pass through the center $\left(\frac{3+5}{2}, \frac{5+9}{2}\right) = (4, 7)$, so its slope is

$$m = \frac{\Delta y}{\Delta x} = \frac{7}{4}.$$

3. (a) Solve for x: $\sqrt{3\sqrt{3}} = 3^x$.

Solution: Squaring twice gives $\left(3\sqrt{3}\right)^2 = \left((3^x)^2\right)^2 = 3^{4x} = 3^3$, so $4x = 3$, or $x = \frac{3}{4}$.

Solution: (Alternate) $3^x = \sqrt{3\sqrt{3}} = \sqrt{3^{\frac{3}{2}}} = 3^{\frac{3}{4}}$, so $x = \frac{3}{4}$.

(b) Solve the equation $x^5 - 5x^3 + 4x = 0$.

Solution: $x^5 - 5x^3 + 4x = x(x^4 - 5x^2 + 4) = x(x^2 - 1)(x^2 - 4) = x(x+1)(x-1)(x+2)(x-2) = 0$, so $x = 0, \pm 1$ or ± 2.

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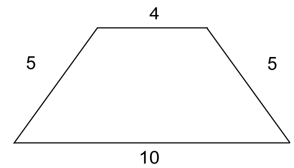
4. (a) An isosceles trapezoid has parallel sides of length 4 and 10, as in the diagram. Find its area.

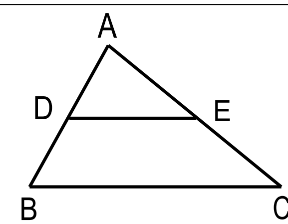


Solution: Vertical lines partition the trapezoid into a rectangle and two right triangles with one side 3 and hypotenuse 5, so other side is 4. The two triangles together have area $3 \cdot 4 = 12$, so the trapezoid has area $4 \times 10 - 12 = 28$. (Or, $A = \frac{b_1 + b_2}{2} \cdot h = \frac{4+10}{2} \cdot 4 = 28$.)

Solution: **(Alternate)** From the top left vertex a line parallel to the right side decomposes the trapezoid into a parallelogram and an isosceles triangle. The latter has height 4 (by Pythagoras' Theorem) and since this is the common height the area is $A = 12 + 16 = 28$.

Solution: **(Note)** Once the height is obtained (e.g., by Pythagoras' Theorem) the standard formula for the area of a trapezoid may also be used: $A = 4\frac{4+10}{2} = 28$.

(b) In $\triangle ABC$, D is the midpoint of AB and E is the midpoint of AC. If $\triangle ADE$ has an area of 4, what is the area of trapezoid $DECB$?



Solution: $\triangle ABC$ is similar to $\triangle ADE$, scaled $2\times$ and so has 4 times the area. Thus the trapezoid has three times the area of $\triangle ADE$, which is 12.

Solution: **(Alternate)** Another approach is to use the fact that the segments joining the midpoints of the three sides partition the triangle into four congruent triangles, three of which form the required area.

5. (a) Given that numbers x, y and z satisfy the equations

$$x + 2y + 3z = 2008 \text{ and } 3x + 2y + z = 8002,$$

what is the value of $x + y + z$?

Solution:

$$x + y + z = \frac{(x + 2y + 3z) + (3x + 2y + z)}{4} = \frac{2008 + 8002}{4} = \frac{10,010}{4} = 2502.5.$$

The answer can also be left in the form $\frac{5005}{2}$.

(b) Solve the equation $(x^2 - 3x + 2)^2 + (x^2 - 4x + 3)^2 + (x^2 - 5x + 4)^2 = 0$.

Solution: The sum of three squares is equal to zero precisely when all three are zero. Therefore,

$$x^2-3x+2 = x^2-4x+3 = x^2-5x+4 = 0 = (x-1)(x-2) = (x-1)(x-3) = (x-1)(x-4).$$

It follows that $x = 1$.

Solution: **(Note)** Some students solved over the complex numbers, obtaining $x = 1, 3 \pm \frac{i\sqrt{6}}{3}$. Full marks for this, of course!

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6. Find the sum of the digits of the number $10^{100} - 10^8 - 3$.

Solution:

$$10^{100} - 10^8 = 10^8 \cdot 99 \cdots 9 = 99 \cdots 900000000$$

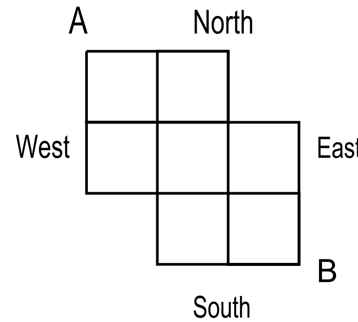
(there are ninety-two 9's). So

$$10^{100} - 10^8 - 3 = 99 \cdots 900000000 - 3 = 99 \cdots 9899999997$$

(the first string of 9's has length 91; the second has length 7). So the sum of the digits is

$$9 \cdot (91 + 7) + 7 + 8 = 9 \cdot 100 - 3 = 897.$$

7. A traveller at A wishes to reach B. To get there he must walk six blocks, travelling only on the streets shown in the diagram. How many possible routes are there?



Solution: Starting at A and moving to the right and down one labels each vertex with the number of paths to arrive at it, which is the sum of the numbers to the left and above it. Thus one arrives, Pascal's-Triangle-style, at the array:

$$\begin{array}{cccc} 1 & 1 & 1 & \\ 1 & 2 & 3 & 3 \\ 1 & 3 & 6 & 9 \\ & 3 & 9 & 18 \end{array}$$

so there are 18 such routes.

Solution: **(Alternate)** Or, one may start with a 3×3 array and subtract the unique paths through the top right and bottom left corners; the standard formula gives the number of paths as

$$\binom{3+3}{3} - 2 = 20 - 2 = 18$$

Solution: **(Alternate)** Even easier: Every path must pass through exactly one of the two vertices, x, y equidistant from A and B. It is easy enough to see that there 3 paths from A to x and so also from x to B, so 9 paths through x; similarly 9 paths through y; thus 18 altogether.

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8. Solve for x, y and z:

$$\begin{aligned} x + y + z &= 4 \\ x - y + z &= 0 \\ x^2 + y^2 + z^2 &= 14 \end{aligned}$$

Solution: The difference between the first two equations is $2y = 4$, or $y = 2$. Eliminating y gives

$$\begin{aligned} x + z &= 2 \\ x^2 + z^2 &= 10 \end{aligned}$$

So we have

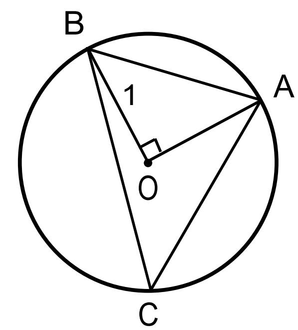
$$(x+z)^2 - (x^2 + z^2) = 2xz = 4 - 10 = -6.$$

Thus $xz = -3$, and x, z are roots of the equation $t^2 - 2t - 3 = (t-3)(t+1) = 0$. So the solutions are

$$(x, y, z) = (-1, 2, 3) \text{ and } (3, 2, -1).$$

9. A, B and C are points on a circle of radius 1 such that $AB = \sqrt{2}$ and $\angle ABC = 60°$. Find AC.

Solution: Let O be the center of the circle. Then $\angle AOB = 90°$. So $\angle ACB = 45°$. The sine law gives



$$\frac{AC}{\sin 60°} = \frac{AB}{\sin 45°}.$$

That is,

$$\frac{AC}{\frac{\sqrt{3}}{2}} = \frac{\sqrt{2}}{\frac{1}{\sqrt{2}}}.$$

So $AC = \sqrt{3}$.

Solution: **(Alternate)** Since $\angle AOC$ stands, at the center, on the same segment as $\angle ABC$, its measure is $120°$. Partitioning it into two $30 - 60 - 90$ triangles one obtains that $AC = 2 \cdot \frac{\sqrt{3}}{2} = \sqrt{3}$. Note that this approach does not require use of sine or cosine laws, and it is seen that the given measure of AB is redundant information.

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10. 2009 points are chosen on the line AB all lying outside the segment AB. Prove that the sum of the distances from these points to the point A is not equal to the sum of their distances to point B.

Solution: Let the points be $C_1, C_2, \ldots, C_{2009}$. Assume that

$$C_1A + \cdots + C_{2009}A = C_1B + \cdots + C_{2009}B.$$

That is,

$$(C_1A - C_1B) + \cdots + (C_{2009}A - C_{2009}B) = 0.$$

But each of $C_iA - C_iB$ is $\pm AB$. So we have $\pm AB \pm \cdots \pm AB = 0$, where there are 2009 terms of equal absolute value AB. So their sum is an odd integer times AB, which cannot be 0, a contradiction. The result follows.