2013 University of Manitoba Open Math Challenge

Are you a "mathlete"? Find out by testing your wits against these problems.

This contest is open to undergraduate students in any UM program. Participation carries no obligation and costs you nothing but your time. You receive your results confidentially except that the overall winner receives **a book prize and $100 cash**, and fame. Interested participants will receive feedback on their own work.

For more details about UM Mathletics visit our web page:

http://server.maths.umanitoba.ca/~craigen/manitobamathletics/

Instructions

PRIOR TO 9 PM OCT. 9, 2013 you must SIGNAL YOUR INTENTION OF COMPETING by sending an email containing your name and student number to **craigenr@cc.umanitoba.ca**. Submit *well-written* solutions **before 2:30 PM, Tuesday, Oct. 15** to R. Craigen (MH 423) or D. Gunderson (MH 421) or, in a sealed envelope to the Math Office (MH 342), or by email to the above address from your university account before the deadline (typeset or high-quality scan of neat handwriting). Solutions will be judged according to correctness, completeness, clarity, elegance, and proper justification. Begin each solution on a new sheet of paper. Staple solutions in the same order as questions are given. You are not expected to solve all questions; submit solutions only for those on which you have made significant progress; do not submit work you consider to be worthless. DO NOT BE DISCOURAGED by these questions—they are not designed to be "easy". Each question is worth 10 marks. *HONOUR SYSTEM: Do not solicit or accept assistance from, or provide it to, others. Do not consult any references or use technology to solve these problems.*

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1. Solve the following equation for x:

$$\frac{(x+2)^2-1}{(x+2)(x-4)+5} = \frac{x+3}{x-3}$$

Solution: The LHS can be re-expressed as $\frac{(x+1)(x+3)}{(x+1)(x-3)}$ and so the equation is a tautology where it is defined. So the solution consists of all real values except $x = 3$ and $x = -1$.

2. A man is walking northbound on Pembina Hwy at 6 km/hr, and notices that in one hour, 40 northbound busses pass him in an hour (as if!). But when he is walking south, he sees 60 northbound busses per hour. Under appropriate assumptions, how fast are the busses going?

Solution: The speed difference between man and bus is proportional to the number of busses seen in a given time interval. If x is the speed of the busses, then $\frac{x-6}{x+6} = \frac{40}{60}$, in which case $x = 30$ km/hr. $\square$

3. Can the numbers $1, 2, \ldots, 2013$ be partitioned into groups so that the largest number in each group is the sum of the other elements in that group?

Solution: Assume that such a partition exists. Then the sum of the numbers in each class is twice the largest element in that class, and so must be an even number. Hence the entire sum $1+2+\cdots+2013$ must also be even, however this sum is $\frac{2014\cdot 2013}{2} = 1007 \cdot 2013$, which is odd. So no such partition exists. $\square$

4. How many rooks can be placed on a $10 \times 10 \times 10$ 3-dimensional chessboard so that no two can attack one another? ("Rooks" on such a "chessboard" attack along lines at a right angle to one of the faces, thus in three different directions.)

Solution: Consider a 10×10 latin square $L = [\ell_{i,j}]$ with entries from $\{1, 2, \ldots, 10\}$. Put a rook in position (i, j, k) iff $\ell_{i,j} = k$.

5. Let Q be any quadrilateral (which may be self-intersecting) in (3 dimensional) space, with vertices A, B, C, D and edges AB, BC, CD and DA. Define a new quadrilateral P whose vertices are the midpoints of the sides of Q and whose edges join midpoints of sides incident on a common endpoint. Prove that P is a parallelogram.

Solution: Let $\vec{v}_0$, $\vec{v}_1$, $\vec{v}_2$, $\vec{v}_3$ be the position vectors for A, B, C, D respectively. The position vectors for the vertices of Q, in order, are thus $\frac{1}{2}(\vec{v}_i + \vec{v}_{i+1})$, $i = 0, 1, 2, 3$ with indices reduced modulo 4. The first and third edges, then, have direction vectors

$$\frac{1}{2}(\vec{v}_0 + \vec{v}_1) - \frac{1}{2}(\vec{v}_1 + \vec{v}_2) = \frac{1}{2}(\vec{v}_0 - \vec{v}_2)$$

and

$$\frac{1}{2}(\vec{v}_2 + \vec{v}_3) - \frac{1}{2}(\vec{v}_3 + \vec{v}_0) = \frac{1}{2}(\vec{v}_2 - \vec{v}_0),$$

which are manifestly dependent; similarly for the other two sides. Opposite sides of P are parallel, so by definition it is a parallelogram (and so also planar – which can alternatively be seen by the fact that the direction vectors for its edges span at most a two-dimensional subspace).

6. Show that, if n is an integer greater than 1, and $2^n + n^2$ is prime, then $n \equiv 3 \pmod 6$. (That is to say, when n is divided by 6, the remainder is 3.)

Solution: First observe that for $n = 3$, get 17, which is prime. Also, when $n = 9$, get 593, which is prime. However when $n = 15$, get 33093, not prime.
If n is even, $2^n + n^2$ is not prime. It remains to show that for any k, $n \neq 6k+1$ and $n \neq 6k+5$. When $n = 6k + 1$,

$$2^n + n^2 = 2^{6k+1} + 36k^2 + 12k + 1$$
$$\equiv (-1)^{6k+1}2 + 1 \pmod 3$$
$$\equiv 0 \pmod 3.$$

Similarly, when $n = 6k + 5$,

$$2^n + n^2 = 2^{6k+5} + 36k^2 + 60k + 25$$
$$\equiv (-1)^{6k+5}2^5 + 1 + 25 \pmod 3$$
$$\equiv 0 \pmod 3.$$

□

7. Two perpendicular lines are tangent to the parabola $y = x^2$. What is the locus of their point of intersection?

Solution: Let such a point be (a, b). Write the two lines as $y - b = m(x - a)$ with m to be determined. The point(s) of tangency are obtained by substituting x^2 for y: $x^2 - b = m(x - a)$ or $x^2 - mx + (b - ma) = 0$ and requiring only one solution. That is, the discriminant is zero:

$$m^2 - 4(b - ma) = 0, \text{ or } m^2 + 4am - 4b = 0.$$

Solving for m we obtain $m = 2(-a \pm \sqrt{a^2 + b^2})$. The condition on (a,b) amounts to saying that the two solutions for m are negative reciprocals:

$$2(-a + \sqrt{a^2 + b^2}) = -1/2(-a - \sqrt{a^2 + b^2}).$$

a is eliminated, and we obtain $b^2 = 1/4$. Eliminating $b = 1/2$, which clearly fails, we obtain the locus $b = -1/2$, or the line $y = -1/2$.
(Not the directrix of the parabola, which is $y = -1/4$.)

8. Given that

$$3\left(1 + \frac{x_1}{2} + \frac{x_2}{4} + \cdots + \frac{x_n}{2^n} + \cdots\right)^2 = 4\left(1 + x_1^2 + x_2^2 + \cdots + x_n^2 + \cdots\right),$$

determine the value of x_{2013}.

Solution: This rearranges into $\left|1 + \sum_{k\geq 1} \frac{x_k}{2^k}\right| = \frac{2}{\sqrt{3}}\sqrt{1^2 + \sum_{k\geq 0} x_k^2}$. It is easy enough to recognize this as the case of equality in the Schwartz inequality, for infinite sequences $(1, x_1, x_2, \ldots)$ and $(1, \frac{1}{2}, \frac{1}{4}, \ldots)$. It follows that the two sequences must be dependent. Comparing first terms shows that they are identical. Thus $x_{2013} = 2^{-2013}$.

9. The *harmonic sum*,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n},$$

is known to grow arbitrarily large as $n \to \infty$. Prove, however, that it is never an integer, except for $n = 1$.

Solution: Let $n \geq 4$. Rewrite the sum as

$$\frac{n!/1 + n!/2 + \cdots + n!/n}{n!}.$$

By Bertrand's postulate, there is a largest prime p in the interval $(n/2, n]$. Then p divides the denominator and all summands except $n!/p$, since $p > n/2$. $\square$

10. Prove that there exist infinitely many primes p with the property that $p^2 - 2$ is not prime.

Solution: Trial and error leads to the discovery that $11^2 - 2 = 119 = 7 \cdot 17$, which is one example and suggests that primes of the form $p = 7k + 4$ are divisible by 7. Indeed, they do, as $(7k + 4)^2 - 2 \equiv 16 - 2 = 14 \equiv 0 \pmod 7$. So it suffices to show that there are infinitely many primes of the form $7k + 4$. Dirichlet's Theorem guarantees the existence of such.