University of Manitoba, Mathletics

Number theory and cases.

Problems from NCS/MAA team competition:

1. (2014Q5) Some positive integers have the interesting property that if you cube the number and add up the (decimal) digits, you get the original number. For example, $8^3 = 512$, and $5+1+2=8$. Also, $17^3 = 4913$, and $4+9+1+3=17$. Are there infinitely many such integers? Defend your answer.

2. (2013Q2) An increasing sequence $3, 15, 24, \ldots$ is formed by the positive multiples of 3 that are one less than a perfect square. Find the 2013th term of the sequence.

3. (2013Q4) Find all pairs (n, k) of positive integers such that the sum of the integers from $n+1$ through $n+k$ inclusive is 2013.

4. (2012Q8) Do there exist positive integers a, b, c, d such that $\frac{a}{b} + \frac{c}{d} = 1$ and $\frac{a}{d} + \frac{c}{b} = 2016$?

5. (2012Q9) Determine all positive integers n such that $(n+1)!$ is divisible by $1! + 2! + \cdots + n!$. (Note that $n! = n \cdot (n-1) \cdots 2 \cdot 1$, for example 2!=2, 5!=120.)

6. (2011Q3) A positive integer k is called "speshul" if there exist positive integers m and n such that

$$\frac{mn+1}{m+n} = k.$$

Find all speshul positive integers.

7. (2011Q4) Determine all pairs (d, r) of positive integers with the property that when each of the numbers 904, 1259 and 2040 is divided by d, the same remainder r (with $0 < r < d$) occurs.

8. (2010Q9) Show that for every positive integer n, $1492^n - 1678^n - 1827^n + 2013^n$ is divisible by 2010.

9. (2009Q10) Prove that there is a positive integer n such that 2009^n in decimal form ends in 000...01, where the final digit 1 is immediately preceded by 2009 zeros.

10. (2007Q6) Find all pairs of positive integers (x, y) with $x \leq y$ such that

$$\sqrt{x} + \sqrt{y} = \sqrt{2007}.$$

11. (2003Q7) Find all pairs (x, y) of integers such that

$$1 + 2002x + 2004y = xy.$$

(We note that 2003 is a prime.)