University of Manitoba, Mathletics
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Inequalities

Facts and definitions

AM-GM inequality: For $a_1, a_2, \dots, a_n \geq 0$,

$$\frac{a_1 + a_2 + \cdots + a_n}{n} \geq (a_1 a_2 \cdots a_n)^{1/n},$$

with equality iff all a_i's are equal.
Cauchy's inequality: For reals $a_1, \dots, a_n, b_1, \dots, b_n$, if $\mathbf{u} = (a_1, \dots, a_n)$ and $\mathbf{v} = (b_1, \dots, b_n)$, then $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \cdot \|\mathbf{v}\|$, or equivalently

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2),$$

where equality holds iff $(a_1, \dots, a_n)$ and $(b_1, \dots, b_n)$ are proportional.
Convex functions, Jensen's inequality: For $x_1, \dots, x_n \in I$, where $I \subset \mathbb{R}$ is an interval, if f is a continuous function on I which is convex (concave up), then

$$f\Big(\frac{x_1 + \ldots + x_n}{n}\Big) \leq \frac{f(x_1) + \ldots + f(x_n)}{n}.$$

If f is strictly convex, then equality holds iff all x_i's are equal.

If the function is concave (concave down), then the sign of the inequality is reversed.

If the function is twice differentiable, it is convex iff $f'' \geq 0$ on the interval. In practice, it is usually easier to show that f' is monotone increasing. The sum of two convex (concave) functions is a convex (concave) function. Multiplication by a positive number also preserves convexity (concavity).
Weighted Jensen's inequality: If $x_1, \dots, x_n \in I$, where I is an interval, $\lambda_1, \dots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$, f is convex on I, then

$$\lambda_1 f(x_1) + \cdots + \lambda_n f(x_n) \geq f(\lambda_1 x_1 + \cdots + \lambda_n x_n).$$

If f is strictly convex, then equality holds iff all x_i's are equal.
Corollary: weighted AM-GM inequality: If $x_1, \dots, x_n \geq 0$, $\lambda_1, \dots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$, then

$$\lambda_1 x_1 + \cdots + \lambda_n x_n \geq x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n},$$

equality holds iff $x_1 = x_2 = \cdots = x_n$.
Power mean inequality: Let $x_1, \dots, x_n \geq 0$, $\lambda_1, \dots, \lambda_n > 0$, $\lambda_1 + \cdots + \lambda_n = 1$. For $t \in \mathbb{R}$, $t \neq 0$, define the weighted mean M_t of order t as $M_t := (\lambda_1 x_1^t + \cdots + \lambda_n x_n^t)^{1/t}$. Also $M_0 := x_1^{\lambda_1} \cdot \ldots \cdot x_n^{\lambda_n} = \lim_{t \to 0} M_t$, $M_{-\infty} := \min\{x_1, \dots, x_n\} = \lim_{t \to -\infty} M_t$, $M_\infty := \max\{x_1, \dots, x_n\} = \lim_{t \to \infty} M_t$. Then

$$M_s \leq M_t, \quad \text{if } -\infty \leq s < t \leq \infty.$$

Problems

1. For $a, b, c \geq 0$, prove $a^3 + b^3 + c^3 \geq a^2 b + b^2 c + c^2 a$.

2. For each positive integer n, find the smallest possible value of the expression $(x_1 + \cdots + x_n)(x_1^{-1} + \cdots + x_n^{-1})$, over all possible $x_1, \ldots, x_n > 0$.

3. If $a, b \geq 0$, $a + b = 2$, prove $(1 + \sqrt[5]{a})^5 + (1 + \sqrt[5]{b})^5 \leq 64$.

4. For $a, b, c > 0$, prove $\dfrac{a^{10} + b^{10} + c^{10}}{a^5 + b^5 + c^5} \geq \left(\dfrac{a + b + c}{3}\right)^5$.

5. For $a_1, \ldots, a_n \geq 0$, prove

$$a_1^5 + \cdots + a_n^5 \geq a_1^3 a_2 a_3 + a_2^3 a_3 a_4 + \cdots + a_n^3 a_1 a_2.$$

6. For $x_1, \ldots, x_n > 0$, prove

$$\frac{x_1^2}{x_1 + x_2} + \frac{x_2^2}{x_2 + x_3} + \cdots + \frac{x_n^2}{x_n + x_1} \geq \frac{x_1 + \cdots + x_n}{2}.$$

7. If $a, b, c > 0$, prove $\dfrac{a}{a + 3b + 3c} + \dfrac{b}{3a + b + 3c} + \dfrac{c}{3a + 3b + c} \geq \dfrac{3}{7}$.

8. If $a_1, \ldots, a_n \geq 1$, prove $\displaystyle\sum_{k=1}^{n} \frac{1}{1 + a_k} \geq \frac{n}{1 + \sqrt[n]{a_1 \cdot \ldots \cdot a_n}}$.

9. For $a_1, \ldots, a_n > 0$ with $a_1 \cdot \ldots \cdot a_n = 1$, prove

$$a_1 + \sqrt{a_2} + \cdots + \sqrt[n]{a_n} \geq \frac{n+1}{2}.$$

10. If $a, b, c > 0$, prove $\left(\dfrac{a + b + c}{3}\right)^{a+b+c} \leq a^a b^b c^c$.

11. For $a_1, \ldots, a_n, b_1, \ldots, b_n \geq 0$, prove

$$((a_1 + b_1) \cdot \ldots \cdot (a_n + b_n))^{\frac{1}{n}} \geq (a_1 \cdot \ldots \cdot a_n)^{\frac{1}{n}} + (b_1 \cdot \ldots \cdot b_n)^{\frac{1}{n}}.$$

12. For $a_1, \ldots, a_n > 0$, prove $a_1^{n+1} + \ldots + a_n^{n+1} \geq a_1 \cdot \ldots \cdot a_n \cdot (a_1 + \ldots + a_n)$.