University of Manitoba, Mathletics
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Integer and Fractional Parts

For a real x, we denote by $\lfloor x \rfloor$ the largest integer not exceeding x (floor function, or integer part). Note that $\lfloor -0.3 \rfloor = -1$. Also, let $\langle x \rangle = x - \lfloor x \rfloor$ be the fractional part of x. We always have $\langle x \rangle \in [0, 1)$.

Problems from NCS/MAA team competition:

1. (2010-1) An integer n is drawn at random from the first 2010 positive integers. What is the probability that $\lfloor \log_3 n \rfloor$ is a multiple of 3?

2. (2006-2) Evaluate $\displaystyle\int_1^{2006} \frac{dx}{x + \lfloor \log_{10} x \rfloor}$.

3. (1999-4) Evaluate $\displaystyle\int_1^{2} \frac{dx}{\lfloor x^2 \rfloor}$.

4. (2001-4) Find a positive number r such that $\langle r \rangle + \left\langle \dfrac{1}{r} \right\rangle = 1$.

5. (2007-4) Evaluate $\displaystyle\int_{-1}^{1} \langle x^2 + 2x - 3 \rangle dx$.

6. (2002-6) Find all integers n such that $\lfloor \sqrt[4]{1} \rfloor + \lfloor \sqrt[4]{2} \rfloor + \lfloor \sqrt[4]{3} \rfloor + \cdots + \lfloor \sqrt[4]{n} \rfloor = 2n$.

7. (2000-7) Find a closed form expression for $f(n) = \displaystyle\sum_{k=1}^{n^2} \frac{n - \lfloor \sqrt{k-1} \rfloor}{\sqrt{k} + \sqrt{k-1}}$.

8. (2011-7) For each positive integer n, let $a_n = \lfloor (n + \sqrt{19})^2 + 2n + \sqrt{99} \rfloor$. Show that a_n is never the square of an integer.

9. (2011-9) Let n be a positive integer which is not a perfect square. Prove that $\langle \sqrt{n} \rangle + \dfrac{1}{2n} < 1$.

10. (2004-9) Let n be an integer, $n \geq 3$, and let x be a real number such that $\langle x \rangle = \langle x^2 \rangle = \langle x^n \rangle$. Prove that x is an integer.

Problems from PUTNAM:

11. (2005-B1) Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.

12. (2001-B3) For any positive integer n, let $\{n\}$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\{n\}} + 2^{-\{n\}}}{2^n}.$$

13. (1998-B4) Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

14. (1997-B1) Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min(\{\frac{m}{6n}\}, \{\frac{m}{3n}\}).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)