University of Manitoba, Mathletics

Number theory and case study.
Set 2

Problems from NCS/MAA team competition:

1. (2010Q5) Find two distinct pairs (m, n) of positive integers satisfying

$$\sum_{k=n}^{m} \frac{1}{k(k+1)} = \frac{1}{2010}.$$

2. (2007Q5) Find all ordered pairs (a, b) of positive integers such that

$$a^2 - b^2 = 2007.$$

3. (2007Q10) Show that the only lattice points (x, y) on the curve

$$y^2 = x^4 + x^3 + x^2 + x + 1$$

are $(-1, \pm 1)$,$(0, \pm 1)$ and $(3, \pm 11)$. (Lattice points are points with integer coordinates.)

4. (2005Q4) If the number 7^{7^7} is written out in decimal form, what is the last (rightmost) digit? Defend your answer.

5. (2000Q6) Observe that $1^3 = 1$ and $71^3 = 357911$ ends in 2 ones. Does there exist a positive integer n for which n^3, in decimal form, ends in 2000 ones?

Set 1 Hints

1. (2014Q5) Let $10^k \leq n < 10^{k+1}$ for some positive integer $k > 1$. Let $S(n)$ denote the sum of the digits of n. If $n = S(n^3) < 9(3k+3)$, then $10^k < 9(3k+3)$, which leads to contradiction.

2. (2013Q2) Answer: $3020^2 - 1$.

3. (2013Q4) Answer: $(2012, 1); (1005, 2); (669, 3); (332, 6); (177, 11); (80, 22); (44, 33); (2, 61)$.

4. (2012Q8) Prove that any rational solution yields an integer solution.

5. (2012Q9) Prove that if $n > 3$, then

$$(n-1) < \frac{n!}{1! + 2! + \ldots + (n-1)!} < n.$$

6. (2011Q3) Prove that $k \leq m$, $k \leq n$ and

$$(m-k)(n-k) = (k-1)(k+1).$$

7. (2011Q4) Prove that $d = 71$.

8. (2010Q9) Hint: $2010 = 6 \cdot 335$.

9. (2009Q10) Prove that there exist two integers m, n such that

$$10^{2010} | 2009^m - 2009^n.$$

10. (2007Q6) $\sqrt{xy}$ is an integer implies $x = m^2k$ and $y = n^2k$, where k is an integer free of squares. Answer: $(223, 892)$

11. (2003Q7) Prove that $2003(x+y) = (x-1)(y+1)$ or $2003^2 = (x-2004)(y-2002)$. Answer: $(2005, 2003^2+2002), (2003, -2003^2+2002), (4007, 4005), (1, -1), (2003^2 + 2004, 2003), (-2003^2 + 2004, 2001)$.