A short remark regarding Pohozaev type results on general domains assuming finite Morse index

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Abstract

We are interested in nonnegative nontrivial solutions of

$$\left\{ \begin{array}{rcll} -\Delta u &=& u^p & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{1}$$

where $1 < p$ and Ω a bounded smooth domain in $\mathbb{R}^N$ with $3 \le N \le 9$. We show that given a nonnegative integer M there is some large $p(M, \Omega)$ such that the only nonnegative solution u, of Morse index at most M, is $u = 0$.

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1 Introduction and results

We are interested in the non-existence of positive classical solutions of

$$\left\{ \begin{array}{rcll} -\Delta u &=& u^p & \text{in } \Omega, \\ u &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{2}$$

where $p > 1$ and Ω is a bounded smooth domain in $\mathbb{R}^N$ where $N \geq 3$.

Define the critical exponent $p_s = \frac{N+2}{N-2}$ and note that it is related to the critical Sobolev imbedding exponent $2^* := \frac{2N}{N-2} = p_s + 1$. For $1 < p < p_s$ one has that $H_0^1(\Omega)$ is compactly imbedded in $L^{p+1}(\Omega)$ and hence one can show the existence of a positive minimizer of

$$\min_{u \in H_0^1(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

This positive minimizer is a positive solution of (2), see for instance the book [18]. For $p \geq p_s$ $H_0^1(\Omega)$ is no longer compactly imbedded in $L^{p+1}(\Omega)$ and so to find positive solutions of (2) one needs to take other approachs. For $p \geq p_s$ the well known Pohozaev identity [15] shows there are no positive solutions of (2) provided Ω is star shaped. For general domains in the critical/supercritical case, $p \geq p_s$, the existence versus nonexistence of positive solutions of (2) is a very delicate question; see [2, 8, 7, 6, 5, 9, 14, 12, 13, 16, 17].

The question we address is a nonexistence result of positive solutions whose Morse index satisfies a certain bound. Before we state our result we define the Morse index of a smooth solution of (2).

Definition 1. *Suppose u is a nonnegative smooth solution of (2). Define the linear operator $L_u(\phi) := -\Delta\phi - pu^{p-1}\phi$ on $H_0^1(\Omega)$. We define the Morse index of u, written $MI(u)$, to be the number of negative eigenvalues, counting multiplicity, of L_u.*

We will also need to discuss the Morse index of a solution defined on the full space.

Definition 2. *Given a solution v of $-\Delta v = f(v)$ in $\mathbb{R}^N$ define the associated energy*

$$I(\psi) := \frac{1}{2}\int_{\mathbb{R}^N} |\nabla\psi|^2 - f'(v)\psi^2 dx.$$

We define the Morse index of v to be the supremum over the dimensions of the subspaces $X \subset C_c^\infty(\mathbb{R}^N)$ such that $I(\psi) < 0$ for all $\psi \in X\backslash\{0\}$.

These two notions are intimately connected and on a bounded domain the definitions are equivalent. Before stating our result we mention two more works. The first is [11]. In this work the author obtains many Liouville results related to solutions which are *stable outside a compact set.* In particular, after a blow up argument, the author obtains a regularity result for (2). In the case of dimensions relevant to us ($3 \le N \le 10$) and $\frac{N+2}{N-2} < p < \infty$ it is shown that a sequence of smooth solutions $\{u_m\}_m$ of (2) is uniformly bounded if and only if their Morse indices are uniformly bounded.

We now state our result.

Theorem 1. *Let Ω denote any smooth bounded domain in $\mathbb{R}^N$ with $3 \le N \le 9$. Suppose M is some positive integer. Then there is some large exponent $p = p(M, \Omega)$ such that the only nonnegative solution of (2) for $p \ge p(M, \Omega)$ with $MI(u) \le M$ is $u = 0$.*

Proof. Suppose the theorem is not true and hence there is some $p_m \to \infty$ and positive smooth solutions $u_m > 0$ of (2) with $p = p_m$ and we can also suppose that the $MI(u_m)$ is bounded by a uniform constant M. Let $0 < t_m := \max_\Omega u_m = u_m(x_m)$ and set $T_m := \frac{t_m}{p_m}$. Define $r_m > 0$ by $r_m^2 := \frac{1}{p_m t_m^{p_m - 1}}$.

We first show that $r_m \to 0$. Towards a contradiction we suppose that there is a subsequence such that r_m is bounded away from zero. Then there is some C such that $p_m t_m^{p_m-1} \le C$. Now we re-write (2) as

$$L_m(u_m) := -\Delta u_m(x) - c_m(x) u_m(x) = 0 \qquad \Omega \qquad\qquad u_m = 0 \qquad \partial\Omega,$$

where $c_m(x) := u_m(x)^{p_m - 1}$. But note that

$$\|c_m\|_{L^\infty} = \|u_m\|_{L^\infty}^{p_m-1} = t_m^{p_m-1} \le \frac{C}{p_m},$$

and since $p_m \to \infty$ we see that for sufficiently large m that L_m satisfies the maximum principle. Hence, for sufficiently large m, we have $u_m = 0$ which gives us the desired contradiction and hence $r_m \to 0$. For the convenience of the reader we give a proof of the above claim regarding the maximum principle. Define $\mu(L_m)$ to be the first eigenvalue of L_m on $H_0^1(\Omega)$ and let $\mu(-\Delta)$ denote the first eigenvalue of $-\Delta$ on $H_0^1(\Omega)$. Let $\phi \in H_0^1(\Omega)$ with $\|\phi\|_{L^2(\Omega)} = 1$. Then we have

$$\int_\Omega L_m(\phi)\phi dx \ge \int_\Omega \phi^2 \left(\mu(-\Delta) - c_m(x)\right) dx \ge \left(\mu(-\Delta) - \frac{C}{p_m}\right),$$

and hence we see that $\mu(L_m)$ is positive for sufficiently large m and hence L_m satisfies a maximum principle.

We now define the quantity $\delta_m := \delta(x_m) := dist(x_m, \partial\Omega) > 0$. By passing to a subsequence we can assume that one of the following holds:

1. $\frac{\delta_m}{r_m} \to \gamma \in [0, \infty)$, or
2. $\frac{\delta_m}{r_m} \to \infty$.

Case 1. Note here that $p_m \delta_m^2 t_m^{p_m-1} \to \gamma^2 \in [0, \infty)$ and so $\delta_m^2 t_m^{p_m-1} \to 0$. Define the rescaled functions

$$w_m(x) := \frac{u_m(x_m + \delta_m x) - u_m(x_m)}{p_m T_m} \qquad x \in \Omega_m := \{x \in \mathbb{R}^N : x_m + \delta_m x \in \Omega\}.$$

Then w_m satisfies

$$\left\{ \begin{array}{rcll} -\Delta w_m &=& \delta_m^2 t_m^{p_m-1} (1 + w_m)^{p_m} & \text{in } \Omega_m, \\ w_m &=& -1 & \text{on } \partial\Omega_m, \end{array} \right. \tag{3}$$

and $-1 \leq w_m \leq 0$ in Ω_m with $w_m(0) = 0$. We are now interested in the limiting behaviour of Ω_m. Firstly note that since Ω is smooth that Ω_m converges to some shifted half space (which is not the case if Ω is only, say, a Lipschitz domain). After a rotation of coordinates one sees that $\Omega_m \to H := \{x \in \mathbb{R}^N : x_N > -1\}$. Passing to the limit in (3) we obtain some w with $-1 \leq w \leq 0$ in H with $\Delta w = 0$ in H and $w = -1$ on ∂H and $w(0) = 0$. This contradicts the strong maximum principle. See [10] and [3].

Case 2. Define the rescaled functions

$$v_m(x) := \frac{u_m(x_m + r_m x) - u_m(x_m)}{T_m} \qquad x \in \Omega_m := \{x \in \mathbb{R}^N : x_m + r_m x \in \Omega\}.$$

Note that v_m satisfies

$$\left\{ \begin{array}{rcll} -\Delta v_m(x) &=& \left(1 + \frac{v_m(x)}{p_m}\right)^{p_m} =: g_m(x) & \text{in } \Omega_m, \\ v_m &=& -p_m & \text{on } \partial\Omega_m. \end{array} \right. \tag{4}$$

Note that $v_m(0) = 0$ and $-p_m \leq v_m \leq 0$ in Ω_m. Note that the bounds on v_m show that $0 \leq g_m(x) \leq 1$ for $x \in \Omega_m$. In addition note that $\Omega_m \to \mathbb{R}^N$ and by passing to a subsequence we can assume Ω_m are nested. We now define $w_m := -v_m \geq 0$ in Ω_m and so $-\Delta w_m = -g_m$ in Ω_m with $w_m(0) = 0$. For any

$0 < R < \infty$ consider k_R to be the smallest integer such that $B_R \subset\subset \Omega_{k_R}$. Then consider w_m for $m \geq k_R$ restricted to Ω_{k_R}. Then by the Harnack inequality there is some $C = C(R, dist(B_R, \partial\Omega_{k_R}))$ (but independent of m) such that

$$\sup_{B_R} w_m \leq C \inf_{B_R} w_m + C\|g_m\|_{L^N(\Omega_{k_R})} = C\|g_m\|_{L^N(\Omega_{k_R})},$$

since $w_m(0) = 0$. Noting that g_m is bounded by 1 we see that for all $0 < R < \infty$ there is some C_R such that $\sup_{B_R} w_m \leq C_R$ for $m \geq k_R$. Returning to v_m we see that for all $m \geq k_R$ we have $\inf_{B_R}(v_m) \geq -C_R$. Using this bound and a diagonal argument we can pass to the limit to find some $v \leq 0$ with $v(0) = 0$ and

$$-\Delta v = e^v \qquad \text{in } \mathbb{R}^N. \tag{5}$$

We now discuss the Morse index of v_m and v. Suppose the Morse index of v_m is n. Then there exists n strictly negative eigenvalues, $\lambda_1, \lambda_2, ..., \lambda_n$ and associated eigenfunctions $\psi_k(x)$, associated with the linearized operator associated with (4), ie. $(\psi_k(x), \lambda_k)$ satisfy

$$\begin{cases} -\Delta\psi_k(x) &= \left(1 + \frac{v_m(x)}{p_m}\right)^{p_m - 1} \psi_k(x) + \lambda_k \psi_k(x) \quad \text{in } \Omega_m, \\ \psi_k &= 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega_m. \end{cases} \tag{6}$$

Now define $\phi_k(x) = \psi_k(\frac{x - x_m}{r_m})$ for $x \in \Omega$. Then $\phi_k(x)$ satisfies

$$\begin{cases} -\Delta\phi_k(x) &= p_m u_m(x)^{p_m - 1}\phi_k(x) + \frac{\lambda_k}{r_m^2}\phi_k(x) \quad \text{in } \Omega, \\ \phi_k &= 0 \qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \end{cases} \tag{7}$$

for $1 \leq k \leq n$. Hence we can conclude that the Morse index of u_m is at least n and by hypothesis we have $n \leq M$. So we have the Morse index of v_m is bounded by M.

Now we suppose that $X \subset C_c^\infty(\mathbb{R}^N)$ a n dimensional linear subspace such that $I(\psi) < 0$ for all $\psi \in X \backslash \{0\}$ where

$$I(\psi) := \int_{\mathbb{R}^N} |\nabla\psi|^2 - e^v \psi^2 dx,$$

and we suppose that $\{\psi_k : 1 \leq k \leq n\}$ forms a basis for X. Denote $I(\psi_k) := -\sigma_k < 0$. Let m_0 be sufficiently large such that the support of every element of X is contained in Ω_m for all $m \geq m_0$. Let $1 \leq k \leq n$ and $m \geq m_0$. Then we have

$$\begin{aligned} I_m(\psi_k) &:= \int_{\Omega_m} |\nabla \psi_k|^2 - e^{v_m} \psi_k^2 dx \\ &= \int_{\Omega_{m_0}} |\nabla \psi_k|^2 - e^{v_m} \psi_k^2 dx \\ &= \int_{\Omega_{m_0}} |\nabla \psi_k|^2 - e^{v} \psi_k^2 dx \\ &\quad + \int_{\Omega_{m_0}} \left(e^v - e^{v_m}\right) \psi_k^2 dx \\ &= -\sigma_k + \int_{\Omega_{m_0}} \left(e^v - e^{v_m}\right) \psi_k^2 dx. \end{aligned}$$

Using the convergence of $v_m \to v$ we see that for sufficiently large m that $I_m(\psi_k) < 0$ for all $1 \leq k \leq n$. From this we can conclude the Morse index of v_m is at least n and hence $n \leq M$. This shows that the Morse index of v, which satisfies (5), is at most M. But in [4] it was shown there are no solutions of $-\Delta v = e^v$ in $\mathbb{R}^N$ which are stable outside a compact provided $3 \leq N \leq 9$. In particular this shows there are no solutions of finite Morse index, and hence we have the desired contradiction.

□

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