Math7460 Homework 4

November 21, 2019

If you do question 1 you can skip question 2.

QUESTION 1. (Optional analysis question) Let H denote a real Hilbert space with inner product given by (x, y). In this question we prove the Riezs representation theorem. Recall it says, for $F \in H^*$ there is a unique $x \in H$ such that

$$\langle F, y\rangle = (x, y) \quad \forall y \in H.$$

You will need the following weak compactness result. Given $\{x_n\}_n \subset H$ (bounded sequence) then there is some subsequence $\{x_{n_k}\}_k$ and $x \in H$ such that $x_{n_k} \rightharpoonup x$ in H (this notation is weak convergence in H).

Set $E(x) := \frac{\|x\|^2}{2} - \langle F, x\rangle$.

PARTt (i). Let $\{x_n\}_n$ denote a minimizing sequence for E; ie. $E(x_n) \to \inf_H E$. Show the sequence is bounded and hence there is a subsequence x_{n_k} and x such that $x_{n_k} \rightharpoonup x$ in H.

PARTt (ii). Show that a norm is weakly lower semi continuous on a Hilbert space; ie. show if $x_m \rightharpoonup x$ in H that $\|x\| \leq \liminf_m \|x_m\|$ (you will need to use a duality proof). Show the same result holds if we replace the norm with $\|\cdot\|^2$.

PARTt (iii). Using the previous parts show that E obtains its infimum over H.

PART (iv). Show the minimzer from part (iii) is exactly the x from teh statement of Riesz Rep. Theorem.

PART (v). The following result isn't really needed for anything... but it might be useful. Suppose E is as above and $x_m \rightharpoonup x$ in H and $E(x_m) \to \inf_{v\in H} E(v)$. Show that in fact one has $x_m \to x$ in H.

PART (vi). Now we prove the result from part (v) but instead of the explicit E lets take $T : H \to \mathbb{R}$ to be a convex smooth mapping with the additional assumption that for all $x \in H$ there is some $\varepsilon_x > 0$ such that

$$T(y) \geq T(x) + \langle T'(x), y - x\rangle + \varepsilon_x \|y - x\|,$$

for all $y \in H$. If you want you can replace the term $\langle T'(x), y - x\rangle$ with $(z_x, y - x)$ where $z_x \in H$ and this is the inner product. Show that if $x_m \rightharpoonup x$ in H and $T(x_m) \to \inf_H T$ then we have $x_m \to x$ in H.

QUESTION 2. Let $f = f(x)$ denote a nice function on Ω and define

$$E(u) := \int_\Omega (\Delta u)^2 dx - \int_\Omega f u dx.$$

PART (i). Define

$$\mathcal{A} := \{u \in C^4(\overline{\Omega}) : u = \partial_\nu u = 0 \text{ on } \partial\Omega\}.$$

Show if there is some $u \in \mathcal{A}$ such that $E(u) = \inf_{v \in \mathcal{A}} E(v)$ then u solves some pde and find this pde.

PART (ii). Define

$$\mathcal{A} := \{u \in C^4(\overline{\Omega}) : u = 0 \text{ on } \partial\Omega\}.$$

Show if there is some $u \in \mathcal{A}$ such that $E(u) = \inf_{v \in \mathcal{A}} E(v)$ then u solves some pde and find this pde.

For question 2 and 3. Usefull to use the Green's indentity

$$\int_\Omega \{(w\Delta v) - v\Delta w)\} \, dx = \int_{\partial\Omega} \{w\partial_\nu v - v\partial_\nu w\} \, d\sigma(x),$$

for $w, v \in C^2(\overline{\Omega})$.

QUESTION 3. Let f, g be nice funtions defined on Ω and $\partial\Omega$ (respectively) and define

$$E(u) := \frac{1}{2}\int_\Omega |\nabla u|^2 dx - \int_\Omega f u dx - \int_{\partial\Omega} g u.$$

Suppose there is some $u \in \mathcal{A} := C^2(\overline{\Omega})$ such that $E(u) = \inf_{v \in \mathcal{A}} E(v)$. What pde does u satisfy?

Question 4. Suppose u is a smooth solution of $\Delta u = 0$ in $\mathbb{R}^N_+$ with $\partial_\nu u = 0$ on $\partial\mathbb{R}^N_+$. Further assume there is some $1 < \sigma < 2$ and $C > 0$ such that $|u(x)| \le C|x|^\sigma$ on $\mathbb{R}^N_+$. Show $u = 0$.

Question 5. Let Ω denote a bounded domain in $\mathbb{R}^N$. Consider a solution $u = u(x,t)$ of

$$u_t - \Delta u = f(x) \qquad (x,t) \in \Omega \times (0,\infty),$$

with $u(x,0) = \phi(x)$ in Ω (assume $\phi = 0$ near $\partial\Omega$ and ϕ nice) with $u = 0$ on $\partial\Omega \times (0, \infty)$. Let $-\Delta v(x) = f(x) \geq 0$ in Ω with $v = 0$ on $\partial\Omega$ (with $v > 0$). Show $u(x,t) \to v(x)$ in some sense as $t \to \infty$.

Hint. Consider $w(x,t) := u(x,t) - v(x)$ and see what equation w satisfies. Then try and find a subsolution and supersolution of the equation that w satisfies (and where both the subsolution and supersolution converge to zero as $t \to \infty$.) Then you can conclude that w does the same.

There are a couple of ways you can try and build a sub/supersolution. You could try something with $-\Delta\phi_1(x) = \lambda_1\phi_1(x)$ in Ω with $\phi_1 = 0$ on $\partial\Omega$ and where $\phi_1 > 0$ (and maybe normalize such that $\sup_\Omega \phi_1 = 1$. Or maybe you can try and build it up from $\psi(x)$ where $-\Delta\psi(x) = 1$ in Ω with $\psi = 0$ on $\partial\Omega$. (note for both ϕ_1, ψ as defined above you might need to make assumptions on their behaviour near $\partial\Omega$. Similarly you might need to make assumptions on $v(x)$ near $\partial\Omega$. Note all of them satisfy Hopf's lemma; ie. $\min_{x\in\partial\Omega} \partial_\nu H(x) < 0$ (here H is any of ϕ_1, ψ, v).

Lax-Milgrim Theorem...not a question. here we prove a generalization of the Riesz rep theorem; namely the 'Lax-Milgrim Theorem'. Let H denote a real Hilbert space with norm and inner product $\|\cdot\|$ and $(\cdot,\cdot)$. Let B denote a bilinear functional on H such that:
(i) there is some $\beta > 0$ such that $|B(x,y)| \leq \beta\|x\|\|y\|$ for all $x, y \in H$, (continuity)
(ii) there is some $\alpha > 0$ such that $\alpha\|x\|^2 \leq B(x,x)$ for all $x \in H$, (co-erviceness).

Then give $F \in H^*$ there is a unique $x \in H$ such that

$$B(x,y) = \langle F, y\rangle \quad \forall y \in H.$$

If B is an inner product on H then this is just Riesz Representation theorem. So you can view this theorem as a form of Riesz for problems without the needed symmetry. The proof we will use will be a 'continuation argument' (this is probably not the standard proof). The idea is to connect something we know about to something we don't.

Linear algebra version. You don't need to do anything here; this is just to give the idea of the proof with a simpler case. Let $A \in \mathbb{R}^{N \times N}$ such that there is some $\alpha > 0$ such that $(Ax) \cdot x \geq \alpha |x|^2$ for all x.
Goal. Our goal is to solve $Ax = b$; of course the above condition implies A is invertible and hence there is no issue; but lets pretend we don't know this theory. Lets decompose A into a symmetric and skew symmetric piece

$$A = \frac{A + A^T}{2} + \frac{A - A^T}{2} =: D + E.$$

Note that $(Dx) \cdot x \geq \alpha |x|^2$. Recall in class we can solve $Dx = b$ by minimizing

$$x \mapsto \frac{(Dx) \cdot x}{2} - b \cdot x,$$

and with the above conditiions on D we can easily minimize this function. So we now want to connect the symmetric problem with the original; so to do this define

$$A_t := D + tE,$$

and set

$$\mathcal{A} := \{t \in [0, 1] : \forall b \in \mathbb{R}^N \exists x \in \mathbb{R}^N \text{ s.t. } A_t x = b\}.$$

We know $0 \in \mathcal{A}$ and we want to show that $\mathcal{A}$ is open and closed and hence $\mathcal{A} = [0, 1]$ (here we don't need to use the topological result...we can prove directly with last point argument.)

$\mathcal{A}$ is closed. let $t_m \in \mathcal{A}$ and $t_m \to t$. Let $b \in \mathbb{R}^N$ so there is some x_m such that $A_{t_m} x_m = b$. Now get bounds on x_m and then pass to a limit.

$\mathcal{A}$ is open. Let $t_0 \in \mathcal{A}$ and we want to show that for ε (of the correct sign if t_0 is an endpoint) that $t = t_0 + \varepsilon \in \mathcal{A}$. Let $b \in \mathbb{R}^N$ and let $A_{t_0} x_0 = b$. Now look for a solution of $A_{t_0+\varepsilon} x = b$ of the form $x = x_0 + z$ (so z is the unkown now). Then we see x is a solution exactly when

$$A_{t_0} z = -\varepsilon E x_0 - \varepsilon E z.$$

To solve this (using a method that extends easily to the infinite dimensinal case) we solve this using a fixed point argument. Given z let $\hat{z}$ solve

$$A_{t_0} \hat{z} = -\varepsilon E x_0 - \varepsilon E z.$$

This defines a mapping $T : \mathbb{R}^N \to \mathbb{R}^N$ by $T(z) = \hat{z}$. So to solve the problem we want to show that T has a fixed point; to do this one can apply the Banach Fixed point theorem.

PART (i) Try and prove the Lax Milgrim theorem using the above outlined approach.