Math7460 Homework 3

November 5, 2019

If you do Question 1 you can skip questions 4,6 and 7.

Question 1. (Optional question; on distributions or "generalized functions") In this question we discuss distributions which are some sort of generalized functions. Just as signed measures can be viewed as some sort of generalization of functions; one can view distributions as even more general. For pde's we mostly care about distributions since it will allow us to make sense of derivatives of functions which are not differentiable in teh classical sense.

Recall if Ω a domain in $\mathbb{R}^N$ (so open set) then $C_c^\infty(\Omega)$ is the set of smooth functions defined on Ω and whose support is contained in Ω (note that if $\phi \in C_c^\infty(\Omega)$ then $\phi = 0$ near $\partial\Omega$).

Definition 1. *(Convergence in $C_c^\infty(\Omega)$). We say that $\phi_m \to \phi$ in $C_c^\infty(\Omega)$ provided: $\phi_m, \phi \in C_c^\infty(\Omega)$ and there is some fixed compact set $K \subset \Omega$ such that for all m we have $supp(\phi_m), supp(\phi) \subset K$ and $\phi_m \to \phi$ in $C^k(K)$ for all $k = 0, 1, 2, \ldots$.*

Definition 2. *(Distributions) Let $T : C_c^\infty(\Omega) \to \mathbb{R}$ denote a linear function (ie. $T(au+bv) = aT(u)+bT(v)$ for all $a, b \in \mathbb{R}$ and $u, v \in C_c^\infty(\Omega)$). We say $T \in \mathcal{D}'(\Omega)$, read T is a* distribution *on Ω, provided T is continuous with respect the above notion of convergence, ie.*

$$\phi_m \to \phi \quad in\ C_c^\infty(\Omega) \qquad \Longrightarrow \qquad T(\phi_m) \to T(\phi).$$

On occasion we use the the alternate notation $\langle T, \phi\rangle = T(\phi)$.

Note this since this notion of convergence is very strong, it makes it fairly easy for a linear function T defined on $C_c^\infty(\Omega)$ to be a distribution.

PART (i). Given $x_0 \in \Omega$ define T by $T(\phi) := \phi(x_0)$. Show $T \in \mathcal{D}'(\Omega)$. The standard notation for this is $T = \delta_{x_0}$ 'The Dirac Mass at x_0"

PART (ii). Given a function $f \in L^1_{Loc}(\Omega)$ (recall $f \in L^1_{loc}(\Omega)$ means $f \in L^1(K)$ for all compact $K \subset \Omega$) we associate a distribution T_f; where

$$\langle T_f, \phi\rangle := \int_\Omega f(x)\phi(x)dx.$$

Show if $f \in L^1_{loc}(\Omega)$ that $T_f \in \mathcal{D}'(\Omega)$.

PART (iii). Given a signed measure μ with finite total variation on each compact $K \subset \Omega$ one can associated a distribution T_μ by

$$\langle T_\mu, \phi\rangle := \int_\Omega \phi(x)d\mu(x).$$

Show $T_\mu \in \mathcal{D}'(\Omega)$.

We now want to discuss the main point of distributions (at least for us). Its the idea of being able to take a derivative of a distribution. In the following definition we will use 'multi-index' notation. Let me explain the notation. Suppose we are on $\Omega \subset \mathbb{R}^N$. Then $\alpha := (\alpha_1, \alpha_2, ..., \alpha_N)$ where $\alpha_k \in \{0, 1, 2, 3...\}$. Suppose u a smooth function defined on Ω and set $\alpha := (2, 1, 4, 1, 0..., 0)$ then

$$\partial^\alpha u(x) = u_{x_1 x_1 x_2 x_3 x_3 x_3 x_3 x_4}.$$

In the following we give the definition of a derivative of a distribution. This is 100 percent motivated by integration by parts. For instance suppose $f(x)$ is smooth on Ω. Using integration by parts write out $\int_\Omega f_{x_1 x_2} \phi(x) dx$ for $\phi \in C_c^\infty(\Omega)$. One would see

$$\int_\Omega f_{x_1 x_2} \phi(x) dx = (-1)^2 \int_\Omega f(x) \phi_{x_1 x_2} dx.$$

Definition 3. *(Derivative of a distribution) Let $T \in \mathcal{D}'(\Omega)$ and let α denote a multi-index. We define the distributional derivative $\partial_\alpha T$ by*

$$\langle \partial_\alpha T, \phi \rangle := (-1)^{|\alpha|} \langle T, \partial_\alpha \phi \rangle.$$

The following theorem says that if you have a distribution T then you can take as many derivatives as you wish and its still a distribution.

Theorem 1. *Let $T \in \mathcal{D}'(\Omega)$ and let α denote a multi-index. Then $\partial_\alpha T$ is also a distribution.*

PART (iv). Let $\Omega \subset \mathbb{R}^N$ and let $T \in \mathcal{D}'(\Omega)$. Show $T_{x_1} \in \mathcal{D}'(\Omega)$. (I am not using the multi index notation here.) Of course this follows from the above theorem. So essentially i want you to prove the above theorem; but for this specific case.

Recall I said before that you can view distributions as more general then signed measures (and signed measures can be viewed as a generalization of functions). The next part deals with this.

Set up for part (v). Let K denote a compact subset of Ω. Recall if μ a signed measure on K with finite total variation one can view $\mu \in (C(K))^*$, (by this I mean you can view μ as a continuous linear functional on $C(K)$ by

$$\mu(\phi) := \int_K \phi(x) d\mu(x).$$

So if $\phi_m \to \phi$ in $C(K)$ them $\mu(\phi_m) \to \mu(\phi)$ in $\mathbb{R}$. Note this continuous linear function above is exactly the distribution we associated with μ and what we called T_μ. So from here on set T_μ by

$$\langle T_\mu, \phi \rangle := \int_\Omega \phi(x) d\mu(x).$$

PART (v). Show there exists some $T \in \mathcal{D}'(\Omega)$ for which there is NO signed measure μ (with finite total variation on each compact $K \subset \Omega$) such that $T = T_\mu$. Hence after identying functions,m measures and distrubtions as before we see we have

$$\text{functions} \subset \text{signed measures} \subset \text{distributions}$$

with all inclusions proper.
Hint. Suppose we can find a $T \in \mathcal{D}'(\Omega)$ such that T is NOT continuous with respect to uniform convergence. Then the T couldn't have been given by $T = T_\mu$ otherwise it would be continuous with respect to uniform convergence. Try working on $\Omega = (-1, 1)$ and try something at zero.

Question 2. Consider the equation given by $u = u(x,t)$

$$\begin{cases} u_t - u_{xx} &= f \qquad (x,t) \in (0,\pi) \times (0,\infty), \\ u &= 0 \qquad (x,t) \in \partial(0,\pi) \times (0,\infty), \\ u(x,0) &= \phi(x) \qquad x \in (0,\pi) \end{cases} \tag{1}$$

where $0 \le f(x,t) \le 1$ with $f(x,t) = 0$ for x near 0 and π. For ϕ we assume $0 \le \phi \le \pi$ is smooth, and compactly supported in $(0,\pi)$ (ie. $\phi = 0$ near $x = 0$ and $x = \pi$).

By the Maximum Principle we have $0 \le u$ on $(x,t) \in (0,\pi) \times (0,\infty)$. The goal of this question is to try and find an upper bound on u by using the Comparison principle.
EDIT. I need to be more precise about how f is compactly supported (we need a uniform in t statement). So you can assume there is some $\delta > 0$ small such that $f(x,t) = 0$ for $0 \le x \le \delta$ and $\pi - \delta \le x \le \pi$ for all $t \ge 0$.

Definition 4. *We call $v = v(x,t)$ a* super solution *of (1) if*

$$\begin{cases} v_t - v_{xx} &\ge f \qquad (x,t) \in (0,\pi) \times (0,\infty), \\ v &\ge 0 \qquad (x,t) \in \partial(0,\pi) \times (0,\infty), \\ v(x,0) &\ge \phi(x) \qquad x \in (0,\pi). \end{cases} \tag{2}$$

By the comparison principle we then have $v \ge u$ in $(x,t) \in (0,\pi) \times (0,\infty)$.

(i) Set $v(x,t) = C_1 x + C_2 t$ and try and pick C_1, C_2 so you can apply the comparison principle and get an upper estimate like $0 \le u \le v$; (you need to tell me exactly how you are taking C_i).

(ii) The estimate from part (i) is quite bad when t is large. The goal of this part is to obtain a much better estimate for large t.
Try $v(x,t) = g(t)\sin(x)$. Write out what you need g to satisfy so that v is a supersolution. Show u is bounded in $(x,t) \in (0,\pi) \times (0,\infty)$. **Hint.** In the end $g(t)$ should satisfy some ode and g should be bounded. You might want to take some sup's in x when defining what ODE g satisfies.

Question 3. Suppose $u \in C^2(\mathbb{R}^N)$ and $\Delta u(x) = 2N$ in $\mathbb{R}^N$ and $u(x) \ge |x|^2 - \frac{\sin(x_1)(x_1 x_N)}{|x|^2+1} - 100$ in $\mathbb{R}^N$. Show $u(x) = |x|^2$.
EDIT. add the condition that $u(0) = 0$.
Edit # 2. Note I minused 100 from the right hand side.

Question 4. (Change of variables) Suppose Ω an open set in $\mathbb{R}^N$ and consider $\int_\Omega f(x)dx$ ($f : \Omega \to \mathbb{R}$). Consider a one to one and onto mapping $T : \Omega' \to \Omega$; $x = T(z)$. Then one has the change of variables formula

$$\int_\Omega f(x)dx = \int_{\Omega'} f(T(z))|DT(z)|dz.$$

Using this show

$$\int_{|x|<R;\ x\in\mathbb{R}^N} f(x)dx = \int_{|z|<1;\ z\in\mathbb{R}^N} f(Rz)R^N dz.$$

Make sure to define exactly what Ω, Ω', T are.

Question 5. Suppose $u \in C^2(\mathbb{R}^N)$ and $\Delta u(x) = 0$ in $\mathbb{R}^N$ (hence we know by a theorem that $u \in C^\infty(\mathbb{R}^N)$. Suppose there is some $C > 0$ such that $|u(x)| \leq C|x| + C$. Show

$$u(x) = c_0 + a \cdot x$$

for some $c_0 \in \mathbb{R}$ and $a \in \mathbb{R}^N$.
Hint. Its sufficient to show that $u_{x_i x_j} = 0$ for $1 \leq i, j \leq N$.

Question 6. Let u be harmonic in the unit ball in $\mathbb{R}^2$ centred at the origin and suppose $u(1, \theta) = g(\theta)$ where we are using polar co-ordinates. Find $u(0, 0)$.

Question 7. Let $A := \{x \in \mathbb{R}^2 : 1 < |x| < 2\}$. Find a solution u of $\Delta u = 0$ in A with $u(1, \theta) = g(\theta)$ and $u(2, \theta) = h(\theta)$.