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HW #3 (continued)

#6 Find the ~~general~~ solution of

$\Delta u = 0$ in $\Omega = (0,\pi) \times (0,\pi)$

$$\begin{cases} u = g_1 & \text{left} \\ u = g_2 & \text{right} \\ \partial_\nu u = h_1 & \text{bottom} \\ \partial_\nu u = h_2 & \text{top.} \end{cases}$$

$\boxed{\#7}$ Here we investigate

$u = u(x,t)$

$$\begin{cases} u_t = u_{xx} + u^2 & x \in (0,\pi) \\ & 0 < t < T \\ u(x,0) = \phi(x) \geq 0 & x \in (0,\pi) \\ u(0,t) = u(\pi,t) = 0 & t > 0 \end{cases} \qquad (1)$$

Facts

a) By maximum principle we can show $u \geq 0$.

b) If ϕ is "small" then there is a global solution of (1) (ie can take $T = \infty$) &

$$\lim_{t \to \infty} \sup_{x \in (0,\pi)} u = 0$$

(c) If ϕ is "Big" then there is a $0 < T < \infty$ & a smooth solution u of (1) on $x \in (0, \pi)$ & $0 \le t < T$

$$\lim_{t \to T} \sup_{x \in (0,\pi)} u = \infty$$

ie solution blows up at time $t = T$.

we will prove portions of this.

~~Let~~

Let $0 \le \phi$ $\phi \not\equiv 0$ & Assume u is solution of (1) on $(0,\pi) \times (0,T)$.

Set $g(t) := \int_0^{\pi} \sin(x)\, u(x,t)\, dx.$ 4

~~Then~~ $\cancel{g'(t) = \int_0^{\pi} \sin(x)\, u_t\, dx}$.

(i) show

$$g'(t) + g(t) = \int_0^{\pi} \sin(x)\, u(x,t)^2\, dx.$$

(ii) Recall Cauchy-Swartz Says

$$\int_0^{\pi} f(x)\, h(x)\, dx \le \left(\int_0^{\pi} f(x)^2\, dx\right)^{\frac{1}{2}} \left(\int_0^{\pi} g(x)^2\, dx\right)^{\frac{1}{2}}.$$

Using this show

$$\frac{1}{2}\left(\int_0^{\pi}\sin(x)\,u(x,t)\,dx\right)^2 \le \int_0^{\pi}\sin(x)u^2\,dx.$$

Hence

$$g'(t)+g(t) \geqslant \frac{1}{2}(g(t))^2 \qquad 0 \le t < T.$$

(iii) Using integrating factor

show

$$\frac{d}{dt}\left(e^t g(t)\right) \geqslant \frac{e^t (g(t))^2}{2}$$

$$0 \le t \le T$$

Now set

$$h(t) := e^t g(t)$$

So show

(iv) $h'(t)(h(t))^{-2} \geq \dfrac{e^{-t}}{2}$ (*)

$0 \leq t < T$

(Assume $h(t) > 0$ which it is since $U(x,t) > 0$ by maximum principle).

~~[illegible]~~

(v) Using (*) & Assuming $T = \infty$ find a condition on ϕ

(Actually on $h(0)$).

So then this has proven:

if $T = \infty \implies$ $h(0)$ satisfies something

hence if

$h(0)$ doesn't satisfy ... $\implies T < \infty$

ie finite time blow up.