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Homework #2

#1 Find the solution of

$$\begin{cases} U_t = U_x + U_{xx} & x \in (0, \pi) \\ & t > 0 \\ U(0,t) = U(\pi,t) = 0 & t > 0 \\ U(x,0) = \varphi(x) & x \in (0, \pi). \end{cases}$$

Question 2

From class

$$u(x,t) = \sum_{k=1}^{\infty} c_k e^{-k^2 t} \sin(kx)$$

$$c_k = \frac{2}{\pi} \int_0^{\pi} \phi(x) \sin(kx)\, dx$$

solves

$$\begin{cases} u_t = u_{xx} & x \in (0,\pi) \\ & t > 0 \\ u(x,0) = \phi(x) & x \in (0,\pi) \\ u(0,t) = u(\pi,t) = 0 & t > 0. \end{cases}$$

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$$\begin{cases} u_t = u_{xx} & x \in (0,\pi) \\ & t > 0 \\ u(x,0) = \phi(x) & x \in (0,\pi) \\ u(0,t) = u(\pi,t) = 0 & t > 0. \end{cases}$$

②

Here we justify the B.C.

~~Let~~ Suppose that

$\int_0^\pi |\phi(x)|dx < \infty.$ Show

for each $t > 0$ that

the series in x

$\sum_{n=1}^\infty c_n e^{-n^2 t} \sin(nx)$ converges

uniformly in x.

③

It is sufficient to show the series is uniformly Cauchy; ie $\forall \varepsilon > 0$ $\exists N_\varepsilon \geq 0$ / (integer) $\forall n \geq m \geq N_\varepsilon$

$$\left| \sum_{k=m}^{n} c_k e^{-k^2 t} \sin(kx) \right| < \varepsilon \cdot / \varphi$$

So for this question show this

$\boxed{\#3}$ ~~Suppose~~

~~(i) Suppose $f \in C^1[0,\pi]$ &~~

~~$f(0) = f(\pi) = 0$~~

let $f \in L^2(0,\pi)$, then ~~can~~

we have

$$\| S_n(f) - f \|_{L^2(0,\pi)} \longrightarrow 0$$

$$S_n(f)(x) = \sum_{n=1}^{n} a_n \sin(nx)$$

where $$a_n = \frac{2}{\pi} \int_0^{\pi} \sin(nx) f(x) dx.$$

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(i) Suppose $f \in C^1[0, \pi]$ & $f(0) = f(\pi) = 0$.

Then

$$a_n = \frac{2}{n\pi} \int_0^\pi f'(x) \cos(nx)\, dx.$$

(ii) If $f \in C^2[0, \pi]$ & $f(0) = f(\pi) = 0$ then

$$a_n = \frac{-2}{n^2 \pi} \int_0^\pi f''(x) \sin(nx)\, dx.$$

(iii) Suppose $f \in C^2[0,\pi]$

$f(0) = f(\pi) = 0$, then

$S_n(f) \to f$ uniformly on $[0,\pi]$.

To do this show $\{S_n(f)\}_n$

is uniform cauchy on $[0,\pi]$

hence $\exists g \in C[0,\pi]$

$S_n(f) \to g$ uniformly on $[0,\pi]$

But $S_n(f) \to f$ in $L^2[0,\pi]$

& show $g(x) = f(x)$ all x.

#3 Boundary Condition fcn

#4 wave eq:

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$$\begin{cases} U_{tt} = U_{xx} & x \in (0, \pi) \\ & t > 0 \\ u(x,0) = \varphi(x) & x \in (0,\pi) \\ U_t(x,0) = f(x) & \\ u(0,t) = u(\pi,t) = 0 & t > 0. \end{cases}$$

From e(a)

$$U(x,t) = \sum_{n=1}^{\infty} \begin{pmatrix} c_k \sin(kt) \\ + d_k \cos(kt) \end{pmatrix} \sin(kx)$$

If c_k & d_k converse fast then infinite sum converges uniformly & we have Boundary condition.

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$$\Phi(x) = \sum_{n=1}^{\infty} d_n \sin(nx)$$

$$\Psi(x) = \sum_{n=1}^{\infty} n C_n \sin(nx)$$

Note since we don't have exponential in time we don't have ~~need~~ converge for earlier argument.

By imposing conditions on Φ & Ψ we can get

...

#4 Gronwall

If $0 \le y \in C^1[0, T)$ &

$y'(t) \le C_1 \, y(t)$ $\left(\begin{array}{c} C_1 \geq 0 \\ \text{or } C_1 < 0 \end{array} \right)$

then

$y(t) \le y(0) \, e^{C_1 t}$ $\forall 0 \le t < T.$

Use integrating factor

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Given $f: \mathbb{R} \to \mathbb{R}$ define
te Fourier transform

$$F(f)(z) = \int_{\mathbb{R}} f(x) e^{-ixz} dz.$$

changed ε to z to
leave ε for other things.

~~(1)~~ note we have complex values. So in what follows

$|x+iy| = |(x,y)|_2$ Euclidean norm

& all L^p spaces are like

$$\|f\|_{L^p(\mathbb{R})} = \left(\int_{\mathbb{R}} |f(x)|^p \, dx \right)^{\frac{1}{p}}. \quad L^{!!}$$

(i) show $F : L^1(\mathbb{R}) \longrightarrow L^\infty(\mathbb{R})$

is a continuous linear

operator. $\underline{\underline{\text{ie}}}$ show its linear

& $\exists\, c > 0$

$$\|F(f)\|_{L^\infty(\mathbb{R})} \le C \|f\|_{L^1(\mathbb{R})}$$

$\forall f \in L^1(\mathbb{R})$.

(ii) If $f \in L^1(\mathbb{R})$

then $F(f) \in C(\mathbb{R})$.

(iii) If $f \in L^1(\mathbb{R})$ then

$F(f) \in C_0(\mathbb{R})$.

$$\left(g \in C_0(\mathbb{R}) \text{ means } g \in C(\mathbb{R}) \text{ \& } \lim_{|z| \to \infty} |g(z)| = 0 \right).$$

Hints

ii) let $f \in L^1(\mathbb{R})$, let $\varepsilon > 0$.

For h small write out

$$F(f)(z+h) - F(f)(z)$$

& ~~break~~ break integral up into piece near $\pm\infty$ & finite part.

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iii) First prove result for $g \in C_c^\infty(\mathbb{R})$ (g smooth & compactly supported).

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let $g \in C_c^\infty(\mathbb{R})$, so $\exists T > 0$

~~$g = 0$ for~~ $|g(x)| = 0$

for $|x| > \frac{T}{2}$.

$$\hat{g}(z) = \int_{\mathbb{R}} g(x) e^{-ixz} \, dx$$

$$= \int_{-T}^{T} g(x) e^{-ixz} \, dx$$

$$= \int_{-T}^{T} g(x) \frac{d}{dx}\left(\frac{e^{-ixz}}{-iz}\right) dx$$

now do integration by parts.

This shows for

$g \in C_c^\infty(\mathbb{R})$

$$\lim_{|z|\to\infty} |\hat{g}(z)| = 0.$$

Now given $f \in L^1(\mathbb{R})$, $\varepsilon > 0$

$\exists\, g_\varepsilon \in C_c^\infty(\mathbb{R})$ s.t.

$\|f - g_\varepsilon\|_{L^1} < \varepsilon$

(you can accept this density result).

Now use fact

$$\hat{f}(z) = (\widehat{f - g_\varepsilon})(z) + (\widehat{g_\varepsilon})(z)$$

ε ...

Actual result

If μ a finite measure

then $\hat{\mu} \in C(\mathbb{R})$.

note δ_0

dirac mass

$$\hat{\delta_0}(z) = \int_{\mathbb{R}} e^{-ixz} \delta_0(x)\, dx$$

$= 1$ is continous 17

but Does not go to zero at $\pm\infty$.