Home work # 2 (continued)

#6 Let $f \in L^2(0,\pi)$ & set

$$S_n(f)(x) := \sum_{k=1}^{n} a_k \sin(kx)$$

where $a_k = \frac{2}{\pi} \int_0^{\pi} f(x) \sin(kx)\,dx$.

we know $\| S_n(f) - f \|_{L^2(0,\pi)} \to 0$

So $\| S_n(f) \|_{L^2}^2$ is Bounded

②

$\&$ so

$$\sum_{n=1}^{n} a_n^2 \frac{\pi}{2} = \|S_n(t)\|_{L^2}^2 \leq C_1$$

$\Rightarrow a_k \to 0$ as $k \to \infty$.

The goal of this question is to prove this $a_n \to 0$ without any knowledge of series using approach similar to question with Fourier transform.

For $n \geq 1$, $f \in L^2(0,\pi)$

define $T_n : L^2(0,\pi) \longrightarrow \mathbb{R}$ by

$$T_n(f) := \int_0^{\pi} f(x)\sin(nx)\,dx.$$

(i). $T_n : L^2(0,\pi) \longrightarrow \mathbb{R}$ is

a "continuous linear functional"

i.e ① its linear

② $\exists\, C_n$ s.t.

$$|T_n(f)| \leq C_n \|f\|_{L^2(0,\pi)}$$

$\forall f \in L^2(0,\pi)$.

Show we can take a

C independent of n s.t.

$$|T_n(f)| \leq C \|f\|_{L^2(0,\pi)} \quad \forall f \in L^2(0,\pi)$$

$\forall n \geq 1.$

Let $f \in L^2(0,\pi)$, $\varepsilon > 0$.

Can Assume $\exists g_\varepsilon \in C_c^\infty(0,\pi)$ s.t.

$\|f - g_\varepsilon\|_{L^2(0,\pi)} < \varepsilon.$ Then

we have

$$T_n(f) = T_n(f - g_\varepsilon) + T_n(g_\varepsilon)$$

$\Rightarrow$

$$|T_n(f)| \le |T_n(f-g_\varepsilon)| + |T_n(g_\varepsilon)|.$$

keep going . . . .

$\boxed{\#7}$ Let $N \geq 3$ & let

$A = \{ x \in \mathbb{R}^N : 1 < |x| < 2 \}$.

Find the solution of

$$\begin{cases} \Delta u = 0 & \text{in } A \\ u(x) = 0 & |x| = 1 \\ u(x) = 7 & |x| = 2. \end{cases}$$

Hint Everything is radial; so try radial formula.

#8) Let $N \geq 3$ &

$\Omega = \{x \in \mathbb{R}^N : |x| > 1\}$.

(i) Show there is a infinite # of solutions of

$$\begin{cases} \Delta u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases} \qquad (1)$$

(ii) ~~Show if the integral let $T \in \mathbb{R}$~~

~~$\lim_{|x| \to \infty} u(x) = 0$~~

let $T \in \mathbb{R}$. Find a solution of (1) with ~~$\lim_{|x|\to\infty} u(x) = 0$~~

$$\lim_{|x| \to \infty} u(x) = T.$$