

A NONLINEAR ELLIPTIC PROBLEM INVOLVING THE GRADIENT ON A HALF SPACE

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ABSTRACT. We consider perturbations of the diffusive Hamilton-Jacobi equation

$$\begin{cases} -\Delta u &= (1 + g(x))|\nabla u|^p & \text{in } \mathbb{R}_+^N, \\ u &= 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases}$$

for $p > 1$. We prove the existence of a classical solution provided $p \in (\frac{4}{3}, 2)$ and g is bounded with uniform radial decay to zero.

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1. INTRODUCTION

In this work we will investigate perturbations of

$$\begin{cases} -\Delta u &= |\nabla u|^p & \text{in } \mathbb{R}_+^N, \\ u &= 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases} \quad (1)$$

where $\mathbb{R}_+^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N, x_N > 0\}$ and $\frac{4}{3} < p < 2$. In particular we are interested in classical nonzero solutions.

Example 1. For $t > 0$ set

$$u_t(x) := \int_0^{x_N} \frac{1}{((p-1)y + t)^{\frac{1}{p-1}}} dy.$$

A computation shows that for $p > 1$, u_t is a classical solution of (1). For $p > 2$ the solution is unbounded when $x_N \rightarrow \infty$ and when $1 < p < 2$ the solution is bounded. Note that this solution has a closed form. Also note that u_t converges to zero as $t \rightarrow \infty$.

A particular perturbation of the above problem will be

$$\begin{cases} -\Delta u &= (1 + g(x))|\nabla u|^p & \text{in } \mathbb{R}_+^N, \\ u &= 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (2)$$

In particular we are interested in nonzero solutions of (2) for sufficient smooth functions g which satisfy needed assumptions. Our approach will be to linearize around u_t to obtain solutions of (2).

We now state our main theorem.

Theorem 1. Suppose $\frac{4}{3} < p < 2$ and g is bounded, Hölder continuous and satisfies

$$\sup_{|x| > R, x_N \geq 0} |g(x)| \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Then there is a nonzero classical solution of (2).

- Remark 1.** (1) *The conditions on g can surely be weakened but our interest was mainly in not making any smallness assumptions on g .*
 (2) *The condition on p may seem somewhat arbitrary but we mention that the restriction $\frac{4}{3} < p < 2$ ensures that $\mu + 1 - \alpha \in (0, 1)$ (see Section 2.0.1) which is needed for the proof of Liouville-type theorems, Propositions 3 and 4, that arose in the blow up analysis.*

1.1. Background. A well studied problem is the existence versus non-existence of positive solutions of the Lane-Emden equation given by

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (3)$$

where $1 < p$ and Ω is a bounded domain in $\mathbb{R}^N$ (where $N \geq 3$) with smooth boundary. In the subcritical case $1 < p < \frac{N+2}{N-2}$ the problem is very well understood and $H_0^1(\Omega)$ solutions are classical solutions; see [31]. In the case of $p \geq \frac{N+2}{N-2}$ there are no classical positive solutions in the case of the domain being star-shaped; see [44]. In the case of non star-shaped domains much less is known; see for instance [14, 21, 22, 23, 43]. In the case of $1 < p < \frac{N}{N-2}$ ultra weak solutions (non H_0^1 solutions) can be shown to be classical solutions. For $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ one cannot use elliptic regularity to show ultra weak solutions are classical. In particular in [39] for a general bounded domain in $\mathbb{R}^N$ they construct singular ultra weak solutions with a prescribed singular set, see the book [42] for more details on this.

We now consider

$$\begin{cases} -\Delta u = |\nabla u|^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \quad (4)$$

where Ω is a bounded domain in $\mathbb{R}^N$. The first point is that it is a non variational equation and hence there are various standard tools which are not available anymore. The case $0 < p < 1$ has been studied in [2]. Some relevant monographs for this work include [32, 28, 47]. Many people have studied boundary blow up versions of (4) where one removes the minus sign in front of the Laplacian; see for instance [35, 48]. See [2, 5, 6, 7, 8, 9, 10, 11, 12, 13, 25, 26, 29, 30, 33, 34, 45, 37, 38, 40, 41] for more results on equations similar to (4). In particular, the interested reader is referred to [40] for recent developments and a bibliography of significant earlier work, where the author studies isolated singularities at 0 of nonnegative solutions of the more general quasilinear equation

$$\Delta u = |x|^\alpha u^p + |x|^\beta |\nabla u|^q \text{ in } \Omega \setminus \{0\},$$

where $\Omega \subset \mathbb{R}^N$ ($N > 2$) is a C^2 bounded domain containing the origin 0, $\alpha > -2$, $\beta > -1$ and $p, q > 1$, and provides a full classification of positive solutions vanishing on $\partial\Omega$ and the removability of isolated singularities.

Let us finally mention that for the whole space case, it was proved in [37] that any classical solution of (4) when $\Omega = \mathbb{R}^N$ with $p > 1$ has to be constant. Also, for the half-space problem (1) in the superquadratic case $p > 2$, it was proved in [27] a Liouville-type classification, or symmetry result, which asserts that any solution $u \in C^2(\mathbb{R}_+^N) \cap C(\overline{\mathbb{R}_+^N})$ has to be one-dimensional, where the result was obtained by using moving planes technique, combined with Bernstein type estimates and a compactness argument. A similar result in the subquadratic case $p \in (1, 2]$ was

proved in [46].

Before outlining our approach we mention that our work is heavily inspired by the works [20, 39, 42, 17, 18, 19, 24]. Many of these works consider variations of $-\Delta u = u^p$ on the full space or an exterior domain. Their approach is to find an approximate solution and then to linearize around the approximate solution to find a true solution. This generally involves a very detailed linear analysis of the linearized operator associated with approximate solution and then one applies a fixed point argument to find a true solution.

This current work continues the theme of examining $-\Delta u = |\nabla u|^p$ (or variations) for singular or classical solutions, see [15, 16, 4, 1, 3].

We also mention the recent work [27] where they examine various results, some of which are Liouville theorems related to (1).

1.2. Outline of approach. First we note that by a scaling argument, instead of finding a nonzero solution of (2), it is sufficient to find a nonzero solution of

$$\begin{cases} -\Delta u(x) &= (1 + g(\lambda x))|\nabla u(x)|^p & \text{in } \mathbb{R}_+^N, \\ u &= 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases} \quad (5)$$

for some $\lambda > 0$. We will look for a solution of (5) of the form $u(x) = u_t(x) + \phi(x)$ (where $t = 1$; but we leave $t > 0$ arbitrary for now) where ϕ is unknown. Then ϕ must satisfy

$$\begin{cases} \widetilde{L}_t(\phi) &= g(\lambda x)|\nabla u_t + \nabla \phi|^p + |\nabla u_t + \nabla \phi|^p \\ &\quad - |\nabla u_t|^p - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla \phi & \text{in } \mathbb{R}_+^N, \\ \phi &= 0 & \text{on } \partial\mathbb{R}_+^N, \end{cases} \quad (6)$$

where the arguments for all the functions are x except for g and where a computation shows that

$$\widetilde{L}_t(\phi) := -\Delta \phi - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla \phi = -\Delta \phi - \frac{p\phi_{x_N}}{(p-1)x_N + t}.$$

We will develop a linear theory for the mapping L_t , a rescaled version of $\widetilde{L}_t$. We will show for all $t > 0$ there is some $C_t > 0$ such that for all $f \in Y$ there is some $\phi \in X$ (see Section 2.0.1 for the definition X and Y) which satisfies $L_t(\phi) = f$ in $\mathbb{R}_+^N$ with $\phi = 0$ on $\partial\mathbb{R}_+^N$. Moreover one has $\|\phi\|_X \leq C_t\|f\|_Y$. Using this we will find a solution of (6) using a fixed point argument. Toward this we define a nonlinear mapping on B_R (the closed ball of radius R centered at the origin in X) by $J_\lambda(\phi) = J_{\lambda,t}(\phi) = \psi$, where

$$\begin{cases} \widetilde{L}_t(\psi) &= g(\lambda x)|\nabla u_t + \nabla \phi|^p + |\nabla u_t + \nabla \phi|^p \\ &\quad - |\nabla u_t|^p - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla \phi & \text{in } \mathbb{R}_+^N, \\ \psi &= 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (7)$$

2. THE LINEAR THEORY

We begin by collecting the various parameters and function spaces for the reader's convenience.

2.0.1. *The parameters, spaces and linear operators.* Let $p \in (\frac{4}{3}, 2)$, $\alpha = \frac{1}{p-1} > 1$, $\gamma = \frac{p}{p-1} > 1$, $\mu = \frac{\gamma}{2}$ (note this implies that $\mu + 1 - \alpha \in (0, 1)$) and $\sigma > 0$ small (chosen small enough so that our solution in the end is a classical solution after applying elliptic regularity). We introduce the norms

$$\begin{aligned}\|\phi\|_X &:= \sup_{0 < x_N \leq 1} |x_N|^\sigma |\nabla \phi(x)| + \sup_{x_N \geq 1} |x_N|^\alpha |\nabla \phi(x)|, \\ \|f\|_Y &:= \sup_{0 < x_N \leq 1} |x_N|^{\sigma+1} |f(x)| + \sup_{x_N \geq 1} |x_N|^{\alpha+1} |f(x)|,\end{aligned}$$

where for $\phi \in X$ we require $\phi = 0$ on $\partial\mathbb{R}_+^N$. The linear operator we deal with is, for $t \geq 1$,

$$L_t(\phi) = \Delta\phi + \frac{\gamma\phi_{x_N}}{x_N + t},$$

and note that

$$\widetilde{L}_t(\phi) = -L_{\frac{t}{p-1}}(\phi).$$

After considering the operator L_t it is natural to consider a slight modification of the space X (call it $\widehat{X}$) whose norm is given by

$$\begin{aligned}\|\phi\|_{\widehat{X}} &:= \sup_{0 < x_N \leq 1} \{ |x_N|^\sigma |\nabla \phi(x)| + |x_N|^{\sigma+1} |\Delta\phi(x)| \} \\ &\quad + \sup_{x_N \geq 1} \{ |x_N|^\alpha |\nabla \phi(x)| + |x_N|^{\alpha+1} |\Delta\phi(x)| \},\end{aligned}$$

so we are defining $\widehat{X} := \{ \phi : \|\phi\|_{\widehat{X}} < \infty \text{ and } \phi = 0 \text{ on } \partial\mathbb{R}_+^N \}$.

We will use a change of variables $\psi(x) = (x_N + t)^\mu \phi(x)$ and set L^t by

$$L^t(\psi) := -\Delta\psi + \frac{\mu(\mu-1)\psi}{(x_N + t)^2}.$$

Then $L_t(\phi) = f$ in $\mathbb{R}_+^N$ if $L^t(\psi) = (x_N + t)^\mu f(x)$ in $\mathbb{R}_+^N$. The natural function spaces for ψ are endowed with the norms

$$\|\psi\|_{X_\psi} := \sup_{0 < x_N < 1} |x_N|^{\sigma-1} |\psi(x)| + \sup_{x_N > 1} |x_N|^{\alpha-1-\mu} |\psi(x)|,$$

where as before we take $\psi = 0$ on $\partial\mathbb{R}_+^N$, the Y_ψ norm is given by

$$\|h\|_{Y_\psi} := \sup_{0 < x_N < 1} |x_N|^{\sigma+1} |h(x)| + \sup_{x_N > 1} |x_N|^{\alpha+1-\mu} |h(x)|.$$

Again it is natural to consider the modified X_ψ norm given by

$$\begin{aligned}\|\psi\|_{\widehat{X}_\psi} &:= \sup_{0 < x_N < 1} \{ |x_N|^{\sigma-1} |\psi(x)| + |x_N|^\sigma |\nabla \psi(x)| + |x_N|^{\sigma+1} |\Delta\psi(x)| \} \\ &\quad + \sup_{x_N > 1} \{ |x_N|^{\alpha-1-\mu} |\psi(x)| + |x_N|^{\alpha-\mu} |\nabla \psi(x)| + |x_N|^{\alpha+1-\mu} |\Delta\psi(x)| \},\end{aligned}$$

where we are imposing $\psi = 0$ on $\partial\mathbb{R}_+^N$.

2.1. The linear theory. We need to consider the following equation

$$\begin{cases} L_t(\phi) &= f(x) & \text{in } \mathbb{R}_+^N, \\ \phi &= 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (8)$$

Theorem 2. *For all $t \geq 1$ there is some $C = C_t$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies (8) and $\|\phi\|_X \leq C\|f\|_Y$.*

Instead of working directly with ϕ we prefer to use a change of variables. If we set $\psi(x) = (x_N + t)^\mu \phi(x)$ and set L^t by

$$L^t(\psi) := -\Delta\psi + \frac{\mu(\mu-1)\psi}{(x_N + t)^2},$$

then it is sufficient to develop a theory for

$$\begin{cases} L^t(\psi) &= h(x) & \text{in } \mathbb{R}_+^N, \\ \psi &= 0 & \text{on } \partial\mathbb{R}_+^N. \end{cases} \quad (9)$$

A computation shows that if ψ satisfies (9) with $h(x) = h_f(x) = -(x_N + t)^\mu f(x)$ then ϕ satisfies (8). The result relating the two problems is given by

Proposition 1. *Suppose there is some $C > 0$ such that for all $h \in Y_\psi$ there is some $\psi \in \widehat{X}_\psi$ that solves (9) and $\|\psi\|_{\widehat{X}_\psi} \leq C\|h\|_{Y_\psi}$. If we set $\phi := (x_N + t)^{-\mu}\psi$ and put $h(x) = h_f(x) = -(x_N + t)^\mu f(x)$, where $f \in Y$ with $\|f\|_Y = 1$, then ϕ satisfies (8) and $\|\phi\|_X \leq C_t$.*

Proof. Let $f \in Y$ with $\|f\|_Y = 1$ and set $h(x) = h_f(x) = -(x_N + t)^\mu f(x)$. Then there is some C_t such that $\|h\|_{Y_\psi} \leq C_t$ and hence there is some $C_{1,t}$ and $\psi \in \widehat{X}_\psi$ which solves (9) and $\|\psi\|_{\widehat{X}_\psi} \leq C_{1,t}$. A direct computation shows that ϕ satisfies the needed equation. Also note that

$$\nabla\phi(x) = \frac{\nabla\psi(x)}{(x_N + t)^\mu} - \frac{\mu e_N \psi(x)}{(x_N + t)^{\mu+1}},$$

where e_N is the N^{th} coordinate vector. Since $\psi \in \widehat{X}_\psi$ one easily sees that $\phi \in X$ and there is some C_1 depending only on t, p, N such that $\|\phi\|_X \leq C_1\|\psi\|_{\widehat{X}_\psi}$. This gives the desired result. $\square$

To prove the needed linear theory for L^t we will use a continuation argument and to start the process we will need some results for Laplacian.

Proposition 2. *Assuming the earlier assumptions on the parameters we have $\Delta : \widehat{X}_\psi \rightarrow Y_\psi$ is a homomorphism.*

Proof. Into. Let $\psi \in \widehat{X}_\psi$ with $\Delta\psi = 0$ in $\mathbb{R}_+^N$. Note that for $0 < x_N < 1$ we have $|\psi(x)| \leq Cx_N^{1-\sigma}$ and so $\psi = 0$ on $\partial\mathbb{R}_+^N$. Let $1 \leq i \leq N-1$ and for any fixed $h \in \mathbb{R} \setminus \{0\}$ set

$$\psi^h(x) = \frac{\psi(x + he_i) - \psi(x)}{h},$$

and note that ψ^h is also harmonic in $\mathbb{R}_+^N$. Note also that there is some C_h such that $|\psi^h(x)| \leq C_h x_N^{1-\sigma}$ for $0 < x_N < 1$. Also for $x_N > 1$ we have

$$|\psi^h(x)| \leq \int_0^1 |\nabla\psi(x + the_i)| dt \leq C|x_N|^{\mu-\alpha},$$

where C is independent of h and also note the exponent $\mu-\alpha$ is negative since $p < 2$. We can extend ψ^h oddly across $x_N = 0$ to see that the extension is harmonic and bounded on $\mathbb{R}^N$ and hence is constant. Taking into account the boundary condition of ψ^h we see $\psi^h = 0$ and hence $\psi(x) = \psi(x_N)$ and recalling ψ is harmonic and the bound near $x_N = 0$ we see that $\psi(x_N) = Ax_N$. Now recalling for $x_N > 1$ we have $|\psi(x_N)| \leq Cx_N^{\mu+1-\alpha}$ and since this exponent is in $(0, 1)$ we get $\psi = 0$.

Onto. We will find a supersolution on finite domains and then pass to the limit. To construct our supersolution we will first consider a one dimensional problem. Firstly consider the one dimensional analogs of the X_ψ and Y_ψ norms (written X_ψ^1, Y_ψ^1) on $(0, \infty)$. For $\tilde{h} \in Y_\psi^1$ we want to find an $\tilde{H} \in X_\psi^1$ which solves

$$-\tilde{H}''(x_N) = \tilde{h}(x_N) \quad \text{for } x_N \in (0, \infty), \quad \text{with } \tilde{H}(0) = 0. \quad (10)$$

A direct computation shows that

$$\tilde{H}(x_N) = \int_0^{x_N} \tau \tilde{h}(\tau) d\tau - x_N \int_\infty^{x_N} \tilde{h}(\tau) d\tau,$$

satisfies (10). Additionally one sees there is some C such that $\|\tilde{H}\|_{X_\psi^1} \leq C\|\tilde{h}\|_{Y_\psi^1}$. Set

$$\tilde{h}_0(x_N) = \frac{\chi_{(0,2)}(x_N)}{x_N^{\sigma+1}} + \frac{\chi_{(1,\infty)}(x_N)}{x_N^{\alpha+1-\mu}}$$

and let $\tilde{H}_0$ denote the corresponding solution as defined above and set $\bar{\psi}(x) = \tilde{H}_0(x_N)$; this will be our supersolution on a truncated domain. Now let $h \in Y_\psi$ with $\|h\|_{Y_\psi} = 1$ and for $R > 1$ (big) and $\varepsilon > 0$ (small) consider $Q_{R,\varepsilon} := B_R \times (\varepsilon, R) \subset \mathbb{R}^{N-1} \times \mathbb{R}$. Let C be from the 1 dimensional problem. Let $\psi_{R,\varepsilon}$ denote a solution of

$$-\Delta \psi_{R,\varepsilon}(x) = h(x) \quad \text{in } Q_{R,\varepsilon} \quad \psi_{R,\varepsilon} = 0 \quad \text{on } \partial Q_{R,\varepsilon}.$$

Then by comparison principle we have $\bar{\psi}(x) \geq \psi_{R,\varepsilon}(x)$ in $Q_{R,\varepsilon}$ and one can argue similarly to get $|\psi_{R,\varepsilon}(x)| \leq \bar{\psi}(x)$ in $Q_{R,\varepsilon}$. Hence there is some $C_1 > 0$ such that for all $R > 1$ and $0 < \varepsilon$ small (and independent of h) we have

$$\sup_{0 < x_N < 1; x \in Q_{R,\varepsilon}} x_N^{\sigma-1} |\psi_{R,\varepsilon}(x)| + \sup_{x_N > 1, x \in Q_{R,\varepsilon}} x_N^{\alpha-1-\mu} |\psi_{R,\varepsilon}(x)| \leq C_1.$$

By taking $\varepsilon = \frac{1}{R}$ and using a diagonal argument and compactness we see that we can pass to the limit to find some ψ such that $-\Delta \psi(x) = h(x)$ in $\mathbb{R}_+^N$. Also by fixing x we can pass to the limit in the quantities in the norm and see that $\psi \in X_\psi$ (hence $\psi = 0$ on $\partial \mathbb{R}_+^N$). Additionally we have $\|\psi\|_{X_\psi} \leq C_1$. A standard argument now gives the desired bound in $\widehat{X}_\psi$; we will include the argument for the sake of the reader.

For $0 < x_N < 1$ consider $\tilde{\psi}(y) := x_N^{-1+\sigma} \psi(x + x_N y)$ for $y \in B_{\frac{1}{4}}$. Fix $N < q < \infty$ and then by local regularity there is some $C = C(q, N)$ such that

$$\|\tilde{\psi}\|_{W^{2,q}(B_{\frac{1}{8}})} \leq C\|\Delta \tilde{\psi}\|_{L^q(B_{\frac{1}{4}})} + C\|\tilde{\psi}\|_{L^q(B_{\frac{1}{4}})}, \quad (11)$$

and note the bounds on h and ψ show that the norms on the right are bounded (independent of x in the allowable range). One can now use the Sobolev imbedding to see that

$$\sup_{B_{\frac{1}{8}}} |\nabla \tilde{\psi}| \leq C_q \|\tilde{\psi}\|_{W^{2,q}(B_{\frac{1}{8}})}$$

and hence we have the gradient bounded; writing this out in terms of ψ gives the desired bound on the gradient of ψ . To get the second order bound we directly use the equation for ψ .

A similar argument gives the desired estimate for $x_N > 1$. Combining these results gives the desired $\widehat{X}_\psi$ bounds. $\square$

Theorem 3. *For all $t \geq 1$ there is some C_t such that for all $h \in Y_\psi$ there is some $\psi \in \widehat{X}_\psi$ such that (9) holds and $\|\psi\|_{\widehat{X}_\psi} \leq C_t \|h\|_{Y_\psi}$.*

Proof. Since $\Delta : \widehat{X}_\psi \rightarrow Y$ is a homomorphism we can use a continuation argument to get the desired result. So towards this we consider

$$L_\tau^t(\psi) := -\Delta\psi + \frac{\tau\mu(\mu-1)\psi}{(x_N + t)^2}.$$

Then note that $(\tau, \psi) \mapsto L_\tau^t(\psi)$ is a continuous mapping from $[0, 1] \times \widehat{X}_\psi$ to Y . So to get the desired result it is sufficient to get estimates on this mapping uniformly in τ . So we suppose the result is false and hence there are sequences $\tau_m \in (0, 1]$, $\psi_m \in \widehat{X}_\psi$ and $h_m \in Y_\psi$ such that $\|\psi_m\|_{\widehat{X}_\psi} = 1$ and $\|h_m\|_{Y_\psi} \rightarrow 0$ and $L_{\tau_m}^t(\psi_m) = h_m$ in $\mathbb{R}_+^N$. We first assume that the zero order term in the norm of ψ_m is bounded away from zero; so after renormalizing we can assume that $\|\psi_m\|_{X_\psi} = 1$ and we still have $\|h_m\|_{Y_\psi} \rightarrow 0$. For ease of notation now we will slightly switch notation; we will write $(x, y) \in \mathbb{R}^{N-1} \times (0, \infty)$ instead of $x \in \mathbb{R}_+^N$.

We consider three cases:

- (i) there is $y^m \rightarrow 0$ such that $(y^m)^{\sigma-1} |\psi_m(x^m, y^m)| \geq \frac{1}{2}$,
- (ii) there is some $y^m \rightarrow \infty$ such that $(y^m)^{\alpha-1-\mu} |\psi_m(x^m, y^m)| \geq \frac{1}{2}$,
- (iii) there is some y^m bounded and bounded away from zero such that $|\psi_m(x^m, y^m)|$ is bounded away from zero.

In all three cases we write $\overline{x^m} = (x^m, y^m)$.

Case (i). Set $\psi^m(z) = (y^m)^{\sigma-1} \psi_m(\overline{x^m} + y^m z)$ for $z_N > -1$. Then $|\psi^m(0)|$ is bounded away from zero and

$$|\psi^m(z)| \leq (1 + z_N)^{1-\sigma} \quad \text{for } 0 < y^m(1 + z_N) < 1,$$

and a computation shows that

$$-\Delta\psi^m(z) + \frac{\tau_m\mu(\mu-1)\psi^m(z)}{(z_N + 1 + (y^m)^{-1}t)^2} = \widehat{h}_m(z) \quad \text{in } z_N > -1,$$

with $\psi^m = 0$ on $z_N = -1$ where $\widehat{h}_m(z) = (y^m)^{\sigma+1} h_m(\overline{x^m} + y^m z)$. Note that

$$|\widehat{h}_m(z)| \leq \frac{\|h_m\|_{Y_\psi}}{(1 + z_N)^{\sigma+1}} \quad 0 < y^m(1 + z_N) < 1,$$

and hence $\widehat{h}_m \rightarrow 0$ uniformly away from $z_N = -1$. By a standard compactness and diagonal argument (and after passing to a subsequence) $\psi^m \rightarrow \psi$ locally in $C_{loc}^{1,\delta}(z_N > -1)$ and ψ satisfies $\Delta\psi(z) = 0$ in $z_N > -1$, $|\psi(0)| \neq 0$, $|\psi(z)| \leq (1 + z_N)^{1-\sigma}$. Using a similar argument as in the proof of the previous proposition we see that we must have $\psi = 0$ which is a contradiction.

Case (ii). Set $\psi^m(z) = (y^m)^{\alpha-1-\mu} \psi_m(\overline{x^m} + y^m z)$ for $z_N > -1$. Then $|\psi^m(0)|$ is bounded away from zero and

$$|\psi^m(z)| \leq (1 + z_N)^{\mu+1-\alpha} \quad \text{for } y^m(1 + z_N) > 1,$$

and recall that $\mu + 1 - \alpha \in (0, 1)$. One should note there is an estimate valid for z_N near -1 but we won't need this. A computation shows that

$$-\Delta\psi^m(z) + \frac{\tau_m\mu(\mu-1)\psi^m(z)}{(z_N+1+(y^m)^{-1}t)^2} = \widehat{h}_m(z) \quad \text{in } z_N > -1,$$

with $\psi^m = 0$ on $z_N = -1$, where $\widehat{h}_m(z) = (y^m)^{\alpha-\mu+1}h_m(\overline{x^m}+y^m z)$. A computation shows that

$$|\widehat{h}_m(z)| \leq \frac{\|h_m\|_{Y_\psi}}{(1+z_N)^{\alpha-\mu+1}}, \quad \text{for } y^m(1+z_N) > 1,$$

and hence $\widehat{h}_m \rightarrow 0$ uniformly away from $z_N = -1$. Again by compactness and a diagonal argument we can assume $\psi^m \rightarrow \psi$ in $C_{loc}^{1,\delta}(z_N > -1)$ and $\tau_m \rightarrow \tau \in [0, 1]$ and ψ satisfies

$$-\Delta\psi(z) + \frac{\tau\mu(\mu-1)\psi(z)}{(z_N+1)^2} = 0 \quad \text{in } z_N > -1, \quad (12)$$

with $|\psi(z)| \leq (1+z_N)^{\mu+1-\alpha}$ for $z_N > -1$ and hence $\psi = 0$ on $z_N = -1$. We can now apply Proposition 4 to get the desired contradiction.

Case (iii). Here we set $\psi^m(z) = \psi_m(\overline{x^m} + y^m z)$ for $z_N > -1$. Then $|\psi^m(0)|$ is bounded away from zero and there is some C (independent of m) such that

$$|\psi^m(z)| \leq C\chi_{(-1,1)}(z_N)(1+z_N)^{1-\sigma} + C\chi_{(0,\infty)}(z_N)(1+z_N)^{1+\mu-\alpha}, \quad (13)$$

for $z_N > -1$. A computation shows that

$$-\Delta\psi^m(z) + \frac{\tau_m\mu(\mu-1)\psi^m(z)}{(1+z_N+(y^m)^{-1}t)^2} = \widehat{h}_m(z) \quad \text{in } z_N > -1,$$

where $\widehat{h}_m(z) = (y^m)^2 h_m(\overline{x^m} + y^m z)$ and $\widehat{h}_m \rightarrow 0$ uniformly away from $z_N = -1$. Using compactness and a diagonal argument we have $\psi^m \rightarrow \psi$ in $C_{loc}^{1,\delta}(z_N > -1)$, hence ψ satisfies

$$-\Delta\psi(z) + \frac{\tau\mu(\mu-1)\psi(z)}{(1+z_N+T)^2} = 0 \quad \text{in } z_N > -1$$

with $T = \frac{t}{(y^\infty)^2}$, where $y^m \rightarrow y^\infty \in (0, \infty)$. Note also that $|\psi(0)| \neq 0$ and ψ also satisfies the pointwise bound for ψ^m given in (13). We can now apply Proposition 3 to get the desired contradiction.

We have proven the desired estimates on $\|\psi_m\|_{X_\psi}$, i.e., $\|\psi_m\|_{X_\psi} \rightarrow 0$. To see that in fact $\|\psi_m\|_{\widehat{X}_\psi} \rightarrow 0$ one can now use a standard scaling argument, see the end of the proof of Proposition 2 for an idea of the needed scaling argument. $\square$

2.2. Liouville theorems. In this section we prove the needed Liouville theorems that arose in the blow up analysis.

Proposition 3. *Let $t > 0$, $\tau \in [0, 1]$ and $\psi \in \widehat{X}_\psi$ be such*

$$-\Delta\psi(x) + \frac{\tau\mu(\mu-1)\psi(x)}{(x_N+t)^2} = 0 \quad \text{in } \mathbb{R}_+^N. \quad (14)$$

Then $\psi = 0$.

Proof. The case of $\tau = 0$ has already been handled since this is just the Laplacian. Again we will switch notation to $x = \bar{x} = (x, y)$. For $1 \leq i \leq N-1$ and $0 < |h| \leq 1$ we consider

$$\psi^h(x, y) = \frac{\psi((x, y) + he_i) - \psi(x, y)}{h},$$

and note that ψ^h satisfies the same equation as ψ . Also note that since $t > 0$ the equation has no singularities in it at $y = 0$ and hence ψ is in fact smooth up to the boundary. Also there is some $C > 0$ (independent of h) such that $|\psi^h(x, y)| \leq Cy^{\mu-\alpha}$ for $x_N > 1$ and note $\mu - \alpha < 0$. Also there is some C_h such that $|\psi^h(x, y)| \leq C_h y^{1-\sigma}$ for $0 < y < 1$ and again we have ψ^h is in fact smooth. Using the above bounds we see that ψ^h is bounded and so if we assume its not identically zero we can then assume (after multiplying by -1 if needed) that $\sup_{\mathbb{R}_+^N} \psi^h = T \in (0, \infty)$. If this is attained at some (x^0, y^0) (with $y^0 \in (0, \infty)$) we get a contradiction via the maximum principle. Hence there must be some (x^m, y^m) such that $\psi^h(x^m, y^m) \rightarrow T$ and note that we must have y^m bounded and bounded away from zero after considering the pointwise bound. For $z_N > -1$ we set $\zeta_m(z) = \psi^h((x^m, y^m) + y^m z)$ and note $\zeta_m(0) \rightarrow T$ and $\zeta_m \leq T$. Also note that

$$|\zeta_m(z)| \leq C(y^m)^{\mu-\alpha}(1+z_N)^{\mu-\alpha} \quad \text{for } y^m(1+z_N) > 1, \quad \text{and}$$

$$|\zeta_m(z)| \leq C_h(y^m)^{1-\sigma}(1+z_N)^{1-\sigma} \quad \text{for } 0 < y^m(1+z_N) < 1.$$

By a compactness and diagonal argument we see there is some ζ such that $\psi_m \rightarrow \zeta$ in $C_{loc}^{1,\delta}(z_N > -1)$ and ζ satisfies

$$-\Delta\zeta(z) + \frac{\tau\mu(\mu-1)\zeta(z)}{(1+z_N + \frac{t}{y^\infty})^2} = 0 \quad \text{in } z_N > -1,$$

where $y^m \rightarrow y^\infty \in (0, \infty)$ and ζ satisfies the same pointwise bounds as ζ_m and hence ζ is nonconstant on $z_N > -1$ and attains its maximum at the origin which contradicts the maximum principle. From this we see that ψ^h is zero and hence $\psi(x) = \psi(x_N)$. Returning to the equation for ψ we see it is now an ode of Euler type and hence has solutions of the form

$$\psi(x_N) = C_1(x_N + t)^{\beta_+(\tau)} + C_2(x_N + t)^{\beta_-(\tau)},$$

where

$$\beta_\pm(\tau) = \frac{1}{2} \pm \frac{\sqrt{1 + 4\tau\mu^2 - 4\tau\mu}}{2}.$$

A computation shows that $\beta'_+(\tau) > 0$ for $\tau \in (0, 1)$ and hence for $\tau \in (0, 1]$ one has $\beta_+(\tau) > \beta_+(0) = 1$. Note that

$$\alpha - 1 - \mu + \beta_+(\tau) > \alpha - 1 - \mu + \beta_+(0) = \alpha - \mu > 0,$$

and hence writing out $\limsup_{x_N \rightarrow \infty} x_N^{\alpha-1-\mu} |\psi(x_N)| \leq C$ gives that $C_1 = 0$. To satisfy the boundary condition one sees they must have $C_2 = 0$ and hence $\psi = 0$. $\square$

Proposition 4. *Suppose $\tau \in [0, 1]$ and ψ satisfies*

$$-\Delta\psi(x) + \frac{\tau\mu(\mu-1)\psi(x)}{x_N^2} = 0 \quad \text{in } \mathbb{R}_+^N, \tag{15}$$

with $|\psi(x)| \leq Cx_N^{\mu+1-\alpha}$ for $x \in \mathbb{R}_+^N$. Then $\psi = 0$.

Proof. The case of $\tau = 0$ is handled in the proof of a previous result. As in the previous proof, for $1 \leq i \leq N-1$ and $0 < |h| \leq 1$, we consider

$$\psi^h(x, y) = \frac{\psi((x, y) + he_i) - \psi(x, y)}{h},$$

and note that ψ^h satisfies the same equation as ψ . Note this time the equation is singular on the boundary.

Also there is some $C > 0$ (independent of h) such that $|\psi^h(x, y)| \leq Cy^{\mu-\alpha}$ for all $y > 0$ and note $\mu - \alpha < 0$. Also there is some C_h such that $|\psi^h(x, y)| \leq C_h y^{\mu-\alpha+1}$ for all $y > 0$ and this exponent is positive. Combining the pointwise estimates we see there is some $\varepsilon > 0$ such that

$$\sup_{(x,y) \in \mathbb{R}_+^N} |\psi^h(x, y)| = \sup_{(x,y) \in \mathbb{R}^{N-1} \times (\varepsilon, \varepsilon^{-1})} |\psi^h(x, y)|.$$

We can argue exactly as in the previous case to see that $\psi(x) = \psi(x_N)$ (we have switched notation back to just $x \in \mathbb{R}_+^N$). So we have

$$\psi(x_N) = C_1 x_N^{\beta_+(\tau)} + C_2 x_N^{\beta_-(\tau)},$$

where $\beta_{\pm}(\tau)$ is from the previous proof. Provided we have both $\beta_+(\tau), \beta_-(\tau)$ different from $\mu + 1 - \alpha$ then by sending $x_N \rightarrow 0, \infty$ we can see $C_1 = C_2 = 0$. From the previous proof we know that $\beta_+(\tau) > \mu + 1 - \alpha$ for $\tau > 0$. By using monotonicity in τ one sees that $\beta_-(\tau) < \mu + 1 - \alpha$ and this gives us the desired result. $\square$

3. THE FIXED POINT ARGUMENT

We now will fix $t = 1$. The following lemma includes some fairly standard inequalities that are needed to prove the nonlinear mapping is a contraction. Note there are no smallness assumptions on the y and z terms. See, for instance, [3, 36] for a proof.

Lemma 1. *Suppose $1 < p \leq 2$. Then there is some $C = C_p$ such that for all vectors $x, y, z \in \mathbb{R}^N$ one has*

$$0 \leq |x + y|^p - |x|^p - p|x|^{p-2}x \cdot y \leq C|y|^p, \quad (16)$$

$$\left| |x + y|^p - p|x|^{p-2}x \cdot y - |x + z|^p + p|x|^{p-2}x \cdot z \right| \leq C(|y|^{p-1} + |z|^{p-1})|y - z|. \quad (17)$$

$$\left| |x + y|^p - |x + z|^p \right| \leq C(|y|^{p-1} + |z|^{p-1} + |x|^{p-1})|y - z|. \quad (18)$$

We will now prove Theorem 1 and for the readers convenience we restate the theorem.

Theorem 1. *Suppose $\frac{4}{3} < p < 2$ and g is bounded, Hölder continuous and satisfies*

$$\sup_{|x| > R, x_N \geq 0} |g(x)| \rightarrow 0 \quad \text{as } R \rightarrow \infty.$$

Then there is a nonzero classical solution of (2).

Proof of Theorem 1. We will show that J_λ is a contraction mapping on B_R as we outlined in the outline. In what follows C is a constant that can change from line to line but is independent of λ and R .

Into. Let $0 < R \leq 1$, $\phi \in B_R$ and let $\psi = J_\lambda(\phi)$. Then ψ satisfies (7) and by the linear theory (Theorem 2) and using (16) we see that

$$\begin{aligned} \|J_\lambda(\phi)\|_X = \|\psi\|_X &\leq C\|g(\lambda x)|\nabla u_t|^p\|_Y + C\|g(\lambda x)|\nabla\phi|^p\|_Y + C\|\nabla u_t + \nabla\phi\|^p \\ &\quad - |\nabla u_t|^p - p|\nabla u_t|^{p-2}\nabla u_t \cdot \nabla\phi\|_Y \\ &\leq C\|g(\lambda x)|\nabla u_t|^p\|_Y + C\|\nabla\phi\|^p\|_Y \end{aligned}$$

since g is bounded. Using the bound on ϕ we see that $\|\nabla\phi\|^p\|_Y \leq CR^p$ and we now examine the other term. So towards this we set

$$I_\lambda^1 := \sup_{0 < x_N < 1} x_N^{\sigma+1} |g(\lambda x)| |\nabla u_t|^p \leq C \sup_{0 < x_N < 1} \frac{x_N^{\sigma+1}}{(x_N + t)^{\alpha p}} |g(\lambda x)|, \quad \text{and}$$

$$I_\lambda^2 := \sup_{x_N > 1} x_N^{\alpha+1} |g(\lambda x)| |\nabla u_t|^p \leq C \sup_{x_N > 1} \frac{x_N^{\alpha+1}}{(x_N + t)^{\alpha p}} |g(\lambda x)|,$$

and note that $\|g(\lambda x)|\nabla u_t|^p\|_Y \leq I_\lambda^1 + I_\lambda^2$. Let $0 < \delta < 1$ (small). Set $A(T) := \sup_{x_N > 0, |x| > T} |g(x)|$ and recall that $A(T) \rightarrow 0$ as $T \rightarrow \infty$. Then

$$\begin{aligned} I_\lambda^1 &\leq C \sup_{0 < x_N < \delta} \frac{x_N^{\sigma+1}}{(x_N + t)^{\alpha p}} |g(\lambda x)| + C \sup_{\delta < x_N < 1} \frac{x_N^{\sigma+1}}{(x_N + t)^{\alpha p}} |g(\lambda x)| \\ &\leq C\delta^{\sigma+1} + CA(\lambda\delta). \end{aligned}$$

Similarly one sees that $I_\lambda^2 \leq CA(\lambda)$. Combining the above results and using the fact that A is monotonic we see that

$$\|J_\lambda(\phi)\|_X \leq C \{R^p + \delta^{\sigma+1} + A(\lambda\delta)\}.$$

So for $J_\lambda(B_R) \subset B_R$ it is sufficient that

$$C \{R^p + \delta^{\sigma+1} + A(\lambda\delta)\} \leq R. \quad (19)$$

Contraction. Let $0 < R \leq 1$, $\phi_i \in B_R$ and $\psi_i = J_\lambda(\phi_i)$, $i = 1, 2$. Writing out the equations for ψ_2 and ψ_1 and taking a difference and using (17) and (18) we arrive at

$$\|J_\lambda(\phi_2) - J_\lambda(\phi_1)\|_X = \|\psi_2 - \psi_1\|_X \leq CH_\lambda + CK_\lambda,$$

where

$$H_\lambda = \|g(\lambda x) \{|\nabla u_t|^{p-1} + |\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1}\} |\nabla\phi_2 - \nabla\phi_1|\|_Y, \quad \text{and}$$

$$K_\lambda = \|\{|\nabla\phi_2|^{p-1} + |\nabla\phi_1|^{p-1}\} |\nabla\phi_2 - \nabla\phi_1|\|_Y.$$

We first estimate K_λ . So using the bound on ϕ_2 one can see

$$\begin{aligned} \sup_{0 < x_N < 1} x_N^{\sigma+1} |\nabla\phi_2|^{p-1} |\nabla\phi_2 - \nabla\phi_1| &\leq R^{p-1} \sup_{0 < x_N < 1} x_N^{\sigma+1-\sigma(p-1)-\sigma} x_N^\sigma |\nabla\phi_2 - \nabla\phi_1| \\ &\leq R^{p-1} \|\phi_2 - \phi_1\|_X \end{aligned}$$

provided $\sigma + 1 - \sigma(p-1) - \sigma \geq 0$, which is satisfied after recalling we are taking $\sigma > 0$ very small. A similar argument shows that

$$\sup_{x_N > 1} x_N^{\alpha+1} |\nabla\phi_2|^{p-1} |\nabla\phi_2 - \nabla\phi_1| \leq R^{p-1} \|\phi_2 - \phi_1\|_X \sup_{x_N > 1} x_N^{\alpha+1-\alpha-\alpha(p-1)},$$

and so here we need the exponent to be less or equal zero and note that this exponent is zero. Combining these two results we see that $K_\lambda \leq CR^{p-1} \|\phi_2 - \phi_1\|_X$.

We now examine the H_λ term.

- First we examine the term $\|g(\lambda x)|\nabla u_t|^{p-1}|\nabla\phi_2 - \nabla\phi_1|\|_Y$. Using an argument as before one has

$$\sup_{0 < x_N < 1} x_N |g(\lambda x)| \leq C\delta + CA(\lambda\delta).$$

A computation shows that

$$\begin{aligned} \sup_{0 < x_N < 1} |g(\lambda x)| |\nabla u_t|^{p-1} |\nabla\phi_2 - \nabla\phi_1| &\leq C \|\phi_2 - \phi_1\|_X \sup_{0 < x_N < 1} x_N |g(\lambda x)| \\ &\leq C(\delta + A(\lambda\delta)) \|\phi_2 - \phi_1\|_X. \end{aligned}$$

We now examine the outer portion of the norm,

$$\begin{aligned} \sup_{x_N > 1} x_N^{\alpha+1} |g(\lambda x)| |\nabla u_t|^{p-1} |\nabla\phi_2 - \nabla\phi_1| &= C \sup_{x_N > 1} \frac{x_N}{x_N + t} |g(\lambda x)| \{x_N^\alpha |\nabla\phi_2 - \nabla\phi_1|\} \\ &\leq CA(\lambda) \|\phi_2 - \phi_1\|_X. \end{aligned}$$

Combining the results gives

$$\|g(\lambda x)|\nabla u_t|^{p-1}|\nabla\phi_2 - \nabla\phi_1|\|_Y \leq C \{\delta + A(\lambda\delta)\} \|\phi_2 - \phi_1\|_X \quad (20)$$

after using monotonicity of A .

- We now examine the term $\|g(\lambda x)|\nabla\phi_2|^{p-1}|\nabla\phi_2 - \nabla\phi_1|\|_Y$. Using the estimate for ϕ_2 one sees that

$$\sup_{0 < x_N < 1} x_N^{\sigma+1} |g(\lambda x)| |\nabla\phi_2|^{p-1} |\nabla\phi_2 - \nabla\phi_1| \leq R^{p-1} \|\phi_2 - \phi_1\|_X \sup_{0 < x_N < 1} x_N^{1-\sigma(p-1)} |g(\lambda x)|.$$

A computation as before shows that

$$\sup_{0 < x_N < 1} x_N^{1-\sigma(p-1)} |g(\lambda x)| \leq C\delta^{1-\sigma(p-1)} + CA(\lambda\delta),$$

and hence

$$\sup_{0 < x_N < 1} x_N^{\sigma+1} |g(\lambda x)| |\nabla\phi_2|^{p-1} |\nabla\phi_2 - \nabla\phi_1| \leq CR^{p-1} \left\{ \delta^{1-\sigma(p-1)} + A(\lambda\delta) \right\} \|\phi_2 - \phi_1\|_X.$$

Similarly the outer portion of the norm gives

$$\begin{aligned} \sup_{x_N > 1} x_N^{\alpha+1} |g(\lambda x)| |\nabla\phi_2|^{p-1} |\nabla\phi_2 - \nabla\phi_1| &\leq R^{p-1} \sup_{x_N > 1} |g(\lambda x)| x_N^\alpha |\nabla\phi_2 - \nabla\phi_1| \\ &\leq R^{p-1} A(\lambda) \|\phi_2 - \phi_1\|_X, \end{aligned}$$

and hence combining these two results gives

$$\|g(\lambda x)|\nabla\phi_2|^{p-1}|\nabla\phi_2 - \nabla\phi_1|\|_Y \leq CR^{p-1} \left\{ \delta^{1-\sigma(p-1)} + A(\lambda\delta) \right\} \|\phi_2 - \phi_1\|_X, \quad (21)$$

where again we have used the monotonicity of A .

Combining with the previous results gives

$$H_\lambda \leq C \left\{ \delta + A(\lambda\delta) + R^{p-1} \left(\delta^{1-\sigma(p-1)} + A(\lambda\delta) \right) \right\} \|\phi_2 - \phi_1\|_X.$$

Combining the estimates for H_λ and K_λ shows that

$$\|J_\lambda(\phi_2) - J_\lambda(\phi_1)\|_X \leq C \left\{ R^{p-1} + \delta + A(\lambda\delta) + R^{p-1} \left(\delta^{1-\sigma(p-1)} + A(\lambda\delta) \right) \right\} \|\phi_2 - \phi_1\|_X.$$

Hence, J_λ is a contraction on B_R provided

$$C \left\{ R^{p-1} + \delta + A(\lambda\delta) + R^{p-1} \left(\delta^{1-\sigma(p-1)} + A(\lambda\delta) \right) \right\} \leq \frac{3}{4}. \quad (22)$$

So for J_λ to be a self-map and contraction mapping on B_R we need both (19) and (22) to hold. To pick the R, δ, λ one first chooses $R > 0$ very small but fixed, then fixes δ very small and finally picks λ very big. Once J_λ is a contraction we can use Banach's Contraction Mapping Principle to see there is a fixed point $\phi \in B_R$ and hence we see that $u(x) = u_t(x) + \phi(x)$ is a solution of (5). Note that u_t is smooth and the gradient of ϕ can have slight blow up at $x_N = 0$; depending on $\sigma > 0$. By taking $\sigma > 0$ very small one can apply elliptic regularity to see that u is a classical solution. To see that u is not identically zero one needs to choose $R > 0$ sufficiently small (relative to $\|u_t\|_X$) and then one sees that $|\nabla u(x)| > 0$ for $x_N > 1$ (for instance). $\square$

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