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# Singular solutions of elliptic equations involving nonlinear gradient terms on perturbations of the ball

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**Abstract**

In this article we obtain positive singular solutions of

$$-\Delta u = |\nabla u|^p \qquad \text{in } \Omega, \qquad u = 0 \text{ on } \partial\Omega, \tag{1}$$

where $\Omega$ is a small $C^2$ perturbation of the unit ball in $\mathbb{R}^N$. For $\frac{N}{N-1} < p < 2$ we prove that if $\Omega$ is a sufficiently small $C^2$ perturbation of the unit ball there exists a singular positive weak solution $u$ of (1). In the case of $p > 2$ we prove a similar result but now the positive weak solution $u$ is contained in $C^{0,\frac{p-2}{p-1}}(\overline{\Omega})$ and yet is not in $C^{0,\frac{p-2}{p-1}+\varepsilon}(\overline{\Omega})$ for any $\varepsilon > 0$.



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## 1. Introduction

In this work we are interested in obtaining positive solutions of

$$\begin{cases} -\Delta u = |\nabla u|^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where $p > 1$ and $\Omega$ is a bounded domain in $\mathbb{R}^N$ with a smooth boundary. We first note by the maximum principle that the only classical solution is the trivial solution $u = 0$; to see this re-write the equation as $-\Delta u - b(x) \cdot \nabla u = 0$ where $b(x) := |\nabla u|^{p-2}\nabla u$ and hence if $b$ is sufficiently regular we can apply the maximum principle. So the only hope of finding a positive solution is to find some sort of singular weak solution. Exactly how singular will depend on the value of the parameter $p > 1$; see Example 1 for details and more discussion on this.

### 1.1. Background

A well studied problem is the existence versus non-existence of positive solutions of the Lane–Emden equation given by

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where $1 < p$ and $\Omega$ is a bounded domain in $\mathbb{R}^N$ (where $N \geq 3$) with smooth boundary. In the subcritical case $1 < p < \frac{N+2}{N-2}$ the problem is very well understood and $H_0^1(\Omega)$ solutions are classical solutions; see [24]. In the case of $p \geq \frac{N+2}{N-2}$ there are no classical positive solutions in the case of the domain being star-shaped; see [36]. In the case of non-star-shaped domains much less is known; see for instance [11,16–18,35]. In the case of $1 < p < \frac{N}{N-2}$ ultra weak solutions (non-$H_0^1$ solutions) can be shown to be classical solutions. For $\frac{N}{N-2} < p < \frac{N+2}{N-2}$ one cannot use elliptic regularity to show ultra weak solutions are classical. In particular in [31] for a general bounded domain in $\mathbb{R}^N$ they construct singular ultra weak solutions with a prescribed singular set. We mention that the weighted Hölder spaces we use in our current work were developed in [31], see also [34].

We now return to (2). The first point is that it is a non-variational equation and hence various standard tools are not available anymore. The case $0 < p < 1$ has been studied in [5]. Some relevant monographs for this work include [21,25,38]. Many people have studied boundary blow up versions of (2) where one removes the minus sign in front of the Laplacian; see for instance [28,39]. See [1–4,6–10,19,20,22,23,26,27,37,29,30,32,33] for more results on equations similar to (2). In particular, the interested reader is referred to P.T. Nguyen [32] for recent developments and a bibliography of significant earlier work, where the author studies isolated singularities at 0 of nonnegative solutions of the more general quasilinear equation

$$\Delta u = |x|^\alpha u^p + |x|^\beta |\nabla u|^q \quad \text{in } \Omega \setminus \{0\},$$

where $\Omega \subset \mathbb{R}^N$ ($N > 2$) is a $C^2$ bounded domain containing the origin 0, $\alpha > -2$, $\beta > -1$ and $p, q > 1$, and provides a full classification of positive solutions vanishing on $\partial\Omega$ and the removability of isolated singularities.

### 1.2. Our approach

Before outlining our approach we mention that our work is motivated by [15,31,34,12–14]. Some of these works deal with a full space or exterior domains; but the linear analysis is still quite similar as compared to what we perform.

We begin by looking at the unit ball for explicit positive radial solutions.

**Example 1.** Let $B_1$ denote the unit ball centered at the origin in $\mathbb{R}^N$ for $N \geq 3$. Define $\alpha = (p-1)(N-1)$. In each of the following cases, there is a continuum family of solutions parameterized by $C$; in the case of $p > \frac{N}{N-1}$ the solutions are distributional solutions on the full ball.

- ($p = 1$) Then

$$u(r) = C \int_r^1 \frac{e^y}{y^{N-1}} dy, \qquad C > 0.$$

  Note that the solution is singular at the origin.
- ($1 < p < \frac{N}{N-1}$) In this case, $\alpha < 1$ and

$$u(r) = \int_r^1 \frac{dy}{(Cy^\alpha - \frac{p-1}{1-\alpha} y)^{1/(p-1)}}, \qquad C > \frac{p-1}{1-\alpha}.$$

  The solution is singular at the origin.
- ($p = \frac{N}{N-1}$) In this case, $\alpha = 1$ and

$$u(r) = \int_r^1 \frac{dy}{(Cy - (p-1) y \ln y)^{1/(p-1)}}, \qquad C > 0.$$

  The solution is singular at the origin.
- ($\frac{N}{N-1} < p < 2$) In this case, $\alpha > 1$ and

$$u(r) = \int_r^1 \frac{dy}{(\frac{p-1}{\alpha-1} y + Cy^\alpha)^{1/(p-1)}}, \qquad C > -\frac{p-1}{\alpha-1}.$$

  Define $\tau = (2-p)/(p-1)$. For the special case $C = 0$,

$$u(r) = \left( \frac{\alpha - 1}{p-1} \right)^{1/(p-1)} \frac{1}{\tau} (r^{-\tau} - 1).$$

  A computation then shows that $u$ is a classical solution of (2) in the case of $\Omega = B_1 \backslash \{0\}$ and note that $u$ is singular at the origin. Also note that $u$ is a suitable weak solution on the full ball; since $p > \frac{N}{N-1}$.

- ($p = 2$) Here $\alpha > 1$. The formula for the explicit solution is the same as above. For the special case $C = 0$ we have

$$u(r) = -(N-2)\ln r.$$

- ($p > 2$) The formula for the explicit solution is the same as the case $\frac{N}{N-1} < p < 2$. Here $\alpha > 1$. Define $\tau = (p-2)/(p-1)$. For the special case $C = 0$,

$$u(r) = \left(\frac{\alpha - 1}{p-1}\right)^{1/(p-1)} \frac{1}{\tau}(1 - r^\tau).$$

  A computation shows $u$ is a weak solution of (2) and note also that $u \in C^{0,\frac{p-2}{p-1}}(\overline{\Omega})$ but that $u$ does not belong in any better zero order Hölder space.

In this article we prove the existence of positive solutions of (2) on domains $\Omega \subset \mathbb{R}^N$ which are small perturbations of $B_1$ and which have the desired singular nature as suggested by the above explicit radial examples.

We write the small perturbations of the domain as $\Omega_t$ where $t > 0$ is small and where $\Omega_0 = B_1$. So our goal is to obtain nontrivial weak solutions of

$$\begin{cases} -\Delta_y u(y) = |\nabla_y u(y)|^p & \text{in } \Omega_t, \\ u = 0 & \text{on } \partial\Omega_t, \end{cases} \tag{4}$$

where $\Omega_t$ is a perturbation of the unit ball in $\mathbb{R}^N$; $\Omega_0 = B_1$ and where $N \geq 3$ and $\frac{N}{N-1} < p < 2$ or $p > 2$. Before carrying on we state our main existence result.

**Theorem 1.** *Suppose $N \geq 3$.*

1. *Suppose $\frac{N}{N-1} < p < 2$. Then for sufficiently small $C^2$ perturbations of the unit ball, say $\Omega_t$, there exists a positive singular weak solution $u$ of (4) which blows up at exactly one point $x_t$ (near the origin) and behaves like $u(x) \approx C|x - x_t|^{\frac{p-2}{p-1}}$ near $x_t$. The proof gives the exact behavior near $x_t$.*
2. *Suppose $p > 2$. Then for sufficiently small $C^2$ perturbations of the unit ball, say $\Omega_t$, there exists a positive weak solution $u$ of (4) with $u \in C^\infty(\Omega_t \backslash \{x_t\})$ and with $u \in C^{0,\frac{p-2}{p-1}}(\overline{\Omega_t})$. In addition $u$ is not in $C^{0,q}(\overline{\Omega_t})$ for any $q > \frac{p-2}{p-1}$.*

We now return to the calculations before we stated our theorem. We now perform a change of variables to reduce the problem on small perturbations of the unit ball to the unit ball; this was taken from [15] where they examine the extremal solution of the Gelfand problem on perturbations of the unit ball. Let $\psi : \overline{B_1} \to \mathbb{R}^N$ be a smooth map and for $t > 0$ define

$$\Omega_t := \{x + t\psi(x) : x \in B_1\}.$$

There is some small $0 < t_0$ such that for all $0 < t < t_0$ one has that $\Omega_t$ is diffeomorphic to the unit ball $B_1$. Let $y = x + t\psi(x)$ for $x \in B_1$ and note there is some $\tilde{\psi}$ smooth such that $x = y + t\tilde{\psi}(y)$ for $y \in \Omega_t$. Given $u(y)$ defined on $y \in \Omega_t$ or $v(x)$ defined on $x \in B_1$ we define the other via

$u(y) = v(x)$. A computation shows that to find positive singular (either singular in $L^\infty$ sense in the first range of $p$ or singular in the Hölder sense in the second range of $p$) solution $u(y)$ of (4) on $\Omega_t$ it is equivalent to finding a positive singular solution $v(x)$ of

$$\begin{cases} -\Delta v - E_t(v) = \left(\sum_{i,j,k=1}^N v_{x_j} v_{x_k} \left\{\delta_{ij} + t\frac{\partial \tilde{\psi}^j}{\partial y_i}\right\}\left\{\delta_{ik} + t\frac{\partial \tilde{\psi}^k}{\partial y_i}\right\}\right)^{\frac{p}{2}} & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1, \end{cases} \tag{5}$$

where $E_t$ is the second order linear differential operator given by

$$E_t(v) := 2t\sum_{i,k} v_{x_i x_k} \partial_{y_i} \tilde{\psi}_k + t\sum_{i,k} v_{x_k} \partial_{y_i y_i} \tilde{\psi}_k + t^2 \sum_{i,j,k} v_{x_j x_k} \partial_{y_i} \tilde{\psi}_j \partial_{y_i} \tilde{\psi}_k,$$

and $\delta_{ij} = 0$ if $i \neq j$ and is 1 otherwise.

We now write the right hand side of (5) as $(H_t)^{\frac{p}{2}}$ and so

$$H_t := \sum_{i,j,k=1}^N v_{x_j} v_{x_k} \left\{\delta_{ij} + t\frac{\partial \tilde{\psi}^j}{\partial y_i}\right\}\left\{\delta_{ik} + t\frac{\partial \tilde{\psi}^k}{\partial y_i}\right\}.$$

We will be a little more precise about $H_t$ than before. Writing out $H_t$ gives (where $a_{ij} := \frac{\partial \tilde{\psi}^j}{\partial y_i}$)

$$\begin{aligned} H_t &= \sum_{k=1}^N v_{x_k}^2 \left(1 + 2t a_{kk} + t^2 \sum_i a_{ik}^2\right) \\ &\quad + \sum_{j,k=1, j\neq k}^N v_{x_j} v_{x_k} \left(t a_{jk} + t a_{kj} + t^2 \sum_i a_{ij} a_{ik}\right) \end{aligned}$$

and note we can write $H_t$ as

$$H_t(v)(x) = |\nabla v(x)|^2 + t(A_1(x)\nabla v(x)) \cdot \nabla v(x) + t^2(A_2(x) \cdot \nabla v(x)) \cdot \nabla v(x)$$

where $A_i(x)$ are some smooth bounded $N \times N$ matrices.

**Assumption on range of $p$.** For clarity of presentation we will now assume that $\frac{N}{N-1} < p < 2$. So in particular we will be looking for a singular (in $L^\infty$ sense) solution. We let $w(x)$ denote the explicit positive singular solution on the unit ball given above; so $w(x) := C_p\left(|x|^{\frac{-(2-p)}{p-1}} - 1\right)$ where $C_p^{p-1} := \frac{N-2-\frac{2-p}{p-1}}{\left(\frac{2-p}{p-1}\right)^{p-1}}$ and for future reference we set $\tau := \frac{2-p}{p-1}$. So with this in mind we will look for solutions of (5) of the form $v(x) = w(x) + \phi(x)$ where $\phi$ is to be determined. Then note that $\phi$ must satisfy

$$\begin{cases} L(\phi) = (H_t(w+\phi, x))^{\frac{p}{2}} - |\nabla w|^p - p|\nabla w|^{p-2}\nabla w \cdot \nabla\phi + E_t(w) + E_t(\phi) & \text{in } B_1, \\ \phi = 0 & \text{on } \partial B_1, \end{cases} \tag{6}$$

where

$$L(\phi) := -\Delta\phi + p\tau^{p-1}C_p^{p-1}\frac{x\cdot\nabla\phi}{|x|^2}.$$

To simplify the calculations we will assume that $H_t(v,x) = |\nabla v|^2 + t(A_1(x)\nabla v)\cdot\nabla v$ but the exact same procedure will work for the non-simplified $H_t$. To find a solution of (6) we will apply the Contraction Mapping Theorem due to Banach and so towards this we define the nonlinear mapping $J_t(\phi) = \psi$ where $\psi$ satisfies

$$\begin{cases} L(\psi) = (H_t(w+\phi,x))^{\frac{p}{2}} - |\nabla w|^p - p|\nabla w|^{p-2}\nabla w\cdot\nabla\phi + E_t(w) + E_t(\phi) & \text{in } B_1, \\ \psi = 0 & \text{on } \partial B_1, \end{cases} \tag{7}$$

of course at this point its not clear this nonlinear mapping $J_t$ is well defined.

We now introduce the weighted Hölder spaces we will use for the fixed point argument; these spaces were introduced in [31], see also the monograph [34].

Fix $0<\alpha<1$ and $0<s<\frac{1}{4}$ define $A_s := \{x\in B_1 : s<|x|<2s\}$ and for $0\le k$ (an integer) define

$$[w]_{k,\alpha,s} := \sum_{j=0}^{k} s^j \sup_{A_s}|\nabla^j w| + s^{k+\alpha}\sup_{x,y\in A_s}\frac{|\nabla^k w(x) - \nabla^k w(y)|}{|x-y|^\alpha}.$$

We now define the norm

$$\|w\|_{C_\nu^{k,\alpha}} := \|u\|_{C^{k,\alpha}(\overline{B_1}\setminus B_{\frac{1}{4}})} + \sup_{s\in(0,\frac{1}{4})} s^{-\nu}[w]_{k,\alpha,s}.$$

The explicit norms we will use will be

$$\|f\|_{C_{\nu-2}^{0,\alpha}} = \|f\|_{C^{0,\alpha}(B_1\setminus B_{\frac{1}{4}})} + \sup_{0<s<\frac{1}{4}} s^{2-\nu}\left(\sup_{A_s}|f| + s^\alpha \sup_{x,y\in A_s}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right).$$

$$\begin{aligned}\|\phi\|_{C_\nu^{2,\alpha}} &= \|\phi\|_{C^{2,\alpha}(B_1\setminus B_{\frac{1}{4}})} \\ &\quad + \sup_{0<s<\frac{1}{4}} s^{-\nu}\Bigg\{\sup_{A_s}|\phi| + s\sup_{A_s}|\nabla\phi| + s^2\sup_{A_s}|D^2\phi| \\ &\quad + s^{2+\alpha}\sup_{x,y\in A_s}\frac{|D^2\phi(x)-D^2\phi(y)|}{|x-y|^\alpha}\Bigg\}.\end{aligned}$$

We put $C_{\nu,D}^{2,\alpha}(\overline{B_1}\setminus\{0\})$ to be the set of functions $\phi\in C_\nu^{2,\alpha}(\overline{B_1}\setminus\{0\})$ with $\phi=0$ on $\partial B_1$. In the end we will end up showing that $J_t : B_r \to B_r$ ($B_r$ the closed ball centered at the origin with radius $r$ in $C_{\nu,D}^{2,\alpha}(\overline{B_1}\setminus\{0\})$) is a contraction mapping.

Of course we will need to understand the mapping properties of the linear mapping $L$ on these spaces; which is the topic of the next section. Towards this end we begin with a result regarding the mapping properties of $\Delta$ on these weighted spaces.

**Theorem** *([34], Corollary 2.3). Let $0 < \alpha < 1$, $N \geq 3$ and let $-(N-2) < \nu < 0$. Then there is some $C > 0$ such that for all $f \in C^{0,\alpha}_{\nu-2}$ there is some unique $\phi \in C^{2,\alpha}_{\nu,D}$ such that $-\Delta\phi = f$ in $B_1$ and $\phi = 0$ on $\partial B_1$. Moreover we have $\|\phi\|_{C^{2,\alpha}_{\nu}} \leq C\|f\|_{C^{0,\alpha}_{\nu-2}}$. In fact $\Delta$ is an isomorphism between the spaces.*

**The specific parameter ranges.** As mentioned above we concentrate on the case of $\frac{N}{N-1} < p < 2$ and hence, unless otherwise stated, we now will always assume

$$N \geq 3, \qquad \frac{N}{N-1} < p < 2, \qquad \tau := \frac{2-p}{p-1}, \qquad \nu := -\tau. \tag{8}$$

In the last section of this article we consider the case of the other range of $p$. We remark that much of the linear theory becomes easier if one can take $\nu < 0$ and with $|\nu|$ small; but since we need it for an explicit value we prefer to just cover that case. Note that under these assumptions we have $\nu \in (-(N-2), 0)$.

### *1.3. The linear theory; $\frac{N}{N-1} < p < 2$*

In this section we consider the solvability of the linear equation given by

$$\begin{cases} L(\phi) = f & \text{in } B_1\backslash\{0\}, \\ \phi = 0 & \text{on } \partial B_1. \end{cases} \tag{9}$$

In particular we would hope to be able to obtain the same theory for $L$ as one has for $\Delta$: there is some $C > 0$ such that for all $f \in C^{0,\alpha}_{\nu-2}$ there is some $\phi \in C^{2,\alpha}_{\nu,D}$ which satisfies (9) and $\|\phi\|_{C^{2,\alpha}_{\nu,D}} \leq C\|f\|_{C^{0,\alpha}_{\nu-2}}$. One approach to obtain these estimates would be to apply the standard continuation argument to connect $L$ to $-\Delta$ via,

$$L_\gamma(\phi) := -\Delta\phi + \gamma p \tau^{p-1} C_p^{p-1} \frac{x \cdot \nabla\phi}{|x|^2},$$

and note $L_0 = -\Delta$ and $L_1 = L$. So to get estimates on $L$ one needs to get estimates on $L_\gamma$ independent of $\gamma$. We are unable to do that directly on these spaces; to fix the problem we will remove the first two modes on the involved function spaces.

For $k \geq 0$ we let $(\Theta_k, \lambda_k)$ denote the $k^{th}$ eigenpair of the Laplace–Beltrami operator, $-\Delta_\theta$ on $S^{N-1}$. So we have $-\Delta_\theta\Theta_k(\theta) = \lambda_k\Theta_k(\theta)$ for $\theta \in S^{N-1}$ and we assume the eigenfunctions are $L^2(S^{N-1})$ normalized. Now recall $\lambda_0 = 0$ (multiplicity zero) and $\lambda_1 = N-1$ (multiplicity $N$) and then $\lambda_2 = 2N$. So given a function $\phi$ defined on $B_1$ we can write $\phi$ as $\phi(x) = \sum_{k=0}^{\infty} a_k(r)\Theta_k(\theta)$ for $r = |x|$ and $\theta = \frac{x}{|x|}$ for suitably chosen $a_k(r)$'s.

**Definition 1.** Let $N, p, \tau, \nu$ be as in (8) and define

$$Y = Y^{\alpha}_{\nu-2} := \left\{ f \in C^{0,\alpha}_{\nu-2}(\overline{B_1}\backslash\{0\}) : f \text{ has no } k = 0, 1 \text{ modes} \right\},$$

$$X = X^{\alpha}_{\nu,D} := \left\{ \phi \in C^{2,\alpha}_{\nu,D}(\overline{B_1}\backslash\{0\}) : \phi \text{ has no } k = 0, 1 \text{ modes} \right\},$$

where by no $k = 0, 1$ modes we mean the function can be represented as $\sum_{k=2}^{\infty} a_k(r)\Theta_k(\theta)$; note we are abusing notation a bit here since the multiplicity of the $k = 1$ mode is $N$. Each subspace inherits the full space's norm.

We now come to one of our main linear results.

**Proposition 1.** *Suppose $N, p, \tau, \nu$ satisfy (8). Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ which satisfies (9) and $\|\phi\|_X \leq C\|f\|_Y$.*

**Lemma 1.** *Let $N, p, \tau, \nu$ be as in (8)*

1. *Then $X$ and $Y$ are closed subspaces in their respective spaces.*
2. *$\Delta : X \to Y$ is one to one and onto, continuous with continuous inverse.*

**Proof.** 1. This holds under pointwise convergence.

2. Look at Corollary 2.5 [34]. Under this assumption on $\nu$ and $N$ we have $\Delta : C^{2,\alpha}_{\nu,D} \to C^{0,\alpha}_{\nu-2}$ is an isomorphism. We just need to check that that there is no interference between $k = 0, 1$ modes and other modes; which is clear. $\square$

We now consider the kernel of $L_\gamma$ in $X$.

**Lemma 2** *(Kernel of $L_\gamma$ in $X$). Suppose $N, p, \tau, \nu$ satisfy (8) and suppose $\gamma \in [0, 1]$. Suppose $\phi \in X$ with $L_\gamma(\phi) = 0$ in $B_1\backslash\{0\}$. Then $\phi = 0$.*

**Proof.** Let $\phi \in X$ with $L_\gamma(\phi) = 0$ in $B_1\backslash\{0\}$. We write $\phi(x) = \sum_{k=2}^{\infty} a_k(r)\Theta_k(\theta)$ and then note we have

$$a_k''(r) + \frac{(N-1-\gamma p(N-2-\tau))}{r} a_k'(r) - \frac{\lambda_k}{r^2} a_k(r) = 0 \quad 0 < r < 1,$$

with $a_k(1) = 0$ for $k \geq 2$. Since the equations are Euler we know there are solutions of the form $a_k(r) = r^\alpha$ where $\alpha$ satisfies

$$\alpha^2 + (N-2-\gamma p(N-2-\tau))\alpha - \lambda_k = 0. \tag{10}$$

We now define the parameter $b := b_\gamma := N - 2 - \gamma p(N - 2 - \tau)$. Note the solutions are given by

$$\alpha_k^- := \frac{-b}{2} - \frac{\sqrt{b^2 + 4\lambda_k}}{2}, \qquad \alpha_k^+ := \frac{-b}{2} + \frac{\sqrt{b^2 + 4\lambda_k}}{2},$$

and hence $a_k(r) = C_k(r^{\alpha_k^+} - r^{\alpha_k^-})$ where we have used $a_k(1) = 0$. Note that $\alpha_k^+ > 0$ for $k \geq 2$. Now note that if we have $\alpha_k^- < \nu = -\tau$ then we must have $a_k = 0$ otherwise $a_k \notin C^{2,\alpha}_\nu$. So we need to check when one has $\alpha_k^- < -\tau$. Note that $\alpha_k^- < -\tau$ is equivalent to $b + \sqrt{b^2 + 8N} > 2\tau$ where we have used the fact that $\lambda_2 = 2N$. Assume this is not true then for some $\gamma \in [0, 1]$ we have $\sqrt{b^2 + 8N} \leq 2\tau - b$. This gives $b^2 + 8N \leq (2\tau - b)^2$, implies $b \leq \tau - \frac{2N}{\tau}$. In particular we must have

$$\min_{\gamma\in[0,1]} b(\gamma) = b(1) = N - 2 - p(N - 2 - \tau) \le \tau - \frac{2N}{\tau}.$$

This leads us to

$$(p-1)(\tau+2) \le N(p-1-\frac{2}{\tau}) = N(p-1)\frac{-p}{2-p} < 0,$$

a contradiction. Since $\lambda_k$ is increasing in $k$ we get the desired result for all $k \ge 2$ and hence we see $a_k = 0$ for all $k \ge 2$ and hence $\phi = 0$. □

We now investigate the kernel of $L_\gamma$ on the full space.

**Lemma 3.** *Suppose $N, p, \tau, \nu$ satisfy (8) and $\gamma \in [0, 1]$. Suppose $\psi \in C^\infty(\mathbb{R}^N\setminus\{0\})$ satisfies*

$$-\Delta\psi(x) + \frac{\gamma p \tau^{p-1} C_p^{p-1} x \cdot \nabla\psi(x)}{|x|^2} = 0 \quad \mathbb{R}^N\setminus\{0\}.$$

*We further assume that there is some $C > 0$ such that $|\psi(x)| \le C|x|^\nu$ and $\psi$ has no $k = 0, 1$ modes. Then $\psi = 0$.*

**Proof.** As before we write $\psi(x) = \sum_{k=2}^\infty a_k(r)\Theta_k(\theta)$ and then $a_k$ satisfies (10) and so for each $k \ge 2$ there is some $C_k, D_k$ such that $a_k(r) = C_k r^{\alpha_k^+} + D_k r^{\alpha_k^-}$. To satisfy the decay condition we see, since $\alpha_k^+ > 0$, we must have $C_k = 0$. So $a_k(r) = D_k r^{\alpha_k^-}$. But recall from the proof of Lemma 2 we have $\alpha_k^- < -\tau = \nu$ and hence we must have $D_k = 0$ otherwise the solution is too singular at the origin to belong to the required space. Hence we have $\psi = 0$. □

**Proof of Proposition 1.** Suppose $N, p, \tau, \nu$ satisfy (8) and then recall we have $\Delta : X \to Y$ is an isomorphism. We would now like to show that $(\gamma, \phi) \mapsto L_\gamma(\phi)$ is a continuous linear mapping from $[0, 1] \times C^{2,\alpha}_{\nu,D}(\overline{B_1}\setminus\{0\}) \to C^{0,\alpha}_{\nu-2}(\overline{B_1}\setminus\{0\})$. To see this we need really only examine the gradient term and for these purposes we set $\hat{L}(\phi)(x) := \frac{x\cdot\nabla\phi(x)}{|x|^2}$. Note we have $\frac{x}{|x|^2} \in C^{1,\alpha}_{-1}$ and by Lemma 6 (see section 2) $\nabla\phi \in C^{1,\alpha}_{\nu-1} \hookrightarrow C^{0,\alpha}_{\nu-2}$, and again using this lemma $\hat{L}_\gamma(\phi) = \frac{x}{|x|^2}.\nabla\phi \in C^{1,\alpha}_{\nu-2} \hookrightarrow C^{0,\alpha}_{\nu-2}$ (continuously) and

$$\|\hat{L}_\gamma(\phi)\|_{C^{0,\alpha}_{\nu-2}} \le \|\frac{x}{|x|^2}\|_{C^{1,\alpha}_{-1}} \|\nabla\phi\|_{C^{1,\alpha}_{\nu-1}} \le C\|\phi\|_{C^{2,\alpha}_\nu},$$

so we have the desired continuity of the gradient term.

We now show that we can replace $C^{2,\alpha}_{\nu,D}, C^{0,\alpha}_{\nu-2}$ with $X, Y$. We really only need to check that given $\phi \in X$ we have $\frac{x\cdot\nabla\phi}{|x|^2} \in Y$. So fix $\phi \in X$; so $\phi(x) = \sum_{k=2}^\infty a_k(r)\Theta_k(\theta)$ and so $\nabla\phi(x)\cdot x = r\sum_{k=2}^\infty a_k'(r)\Theta_k(\theta)$ and hence we have $L_\gamma(\phi) \in Y$. So we have $(\gamma, \phi) \to L_\gamma(\phi)$ is a continuous linear operator from $[0, 1] \times X$ to $Y$. Also note that $L_0 = -\Delta$ is an isomorphism. So if we can show the appropriate bounds on $L_\gamma$ then we'd have $L_1 : X \to Y$ is onto with continuous inverse.

So we suppose we don't have the required result and so there is $\gamma_m \in [0, 1]$, $f_m \in Y$ and $\phi_m \in X$ such that

$$L_m(\phi_m) := L_{\gamma_m}(\phi_m) = f_m \text{ in } B_1, \qquad \phi_m = 0 \text{ on } \partial B_1$$

with $\|f_m\|_Y := \|f_m\|_{C^{0,\alpha}_{\nu-2}} \to 0$ and $\|\phi_m\|_X := \|\phi_m\|_{C^{2,\alpha}_\nu} = 1$. So we have

$$\|f_m\|_{C^{0,\alpha}_{\nu-2}} = \|f_m\|_{C^{0,\alpha}(B_1\setminus B_{\frac{1}{4}})} + \sup_{0<s<\frac{1}{4}} s^{2-\nu}\left(\sup_{A_s}|f_m| + s^\alpha \sup_{x,y\in A_s}\frac{|f(x)-f(y)|}{|x-y|^\alpha}\right) \to 0.$$

Also we have

$$\begin{aligned} 1 = \|\phi_m\|_{C^{2,\alpha}_\nu} &= \|\phi_m\|_{C^{2,\alpha}(B_1\setminus B_{\frac{1}{4}})} \\ &+ \sup_{0<s<\frac{1}{4}} s^{-\nu}\Bigg\{\sup_{A_s}|\phi_m| + s\sup_{A_s}|\nabla\phi_m| + s^2\sup_{A_s}|D^2\phi_m| \\ &+ s^{2+\alpha}\sup_{x,y\in A_s}\frac{|D^2\phi_m(x)-D^2\phi_m(y)|}{|x-y|^\alpha}\Bigg\}. \end{aligned}$$

We now try and obtain a contradiction and to do this we consider a few separate cases:
**Case I.** $\|\phi_m\|_{C^{2,\alpha}(B_1\setminus B_{\frac{1}{4}})}$ is bounded away from zero.
**Case II.** $\|\phi_m\|_{C^{2,\alpha}(B_1\setminus B_{\frac{1}{4}})} \to 0$ but there is some $s_m < \frac{1}{4}$ bounded away from zero such that

$$s_m^{-\nu}\left\{\sup_{A_{s_m}}|\phi_m| + s_m\sup_{A_{s_m}}|\nabla\phi_m| + s_m^2\sup_{A_{s_m}}|D^2\phi_m| + s_m^{2+\alpha}\sup_{x,y\in A_{s_m}}\frac{|D^2\phi_m(x)-D^2\phi_m(y)|}{|x-y|^\alpha}\right\}$$

is bounded away from zero.

**Case III.** There is some $s_m \searrow 0$ such that

$$s_m^{-\nu}\left\{\sup_{A_{s_m}}|\phi_m| + s_m\sup_{A_{s_m}}|\nabla\phi_m| + s_m^2\sup_{A_{s_m}}|D^2\phi_m| + s_m^{2+\alpha}\sup_{x,y\in A_{s_m}}\frac{|D^2\phi_m(x)-D^2\phi_m(y)|}{|x-y|^\alpha}\right\} \to 1.$$

We begin with a result we will use numerous times. Suppose $\tilde\nu \in \mathbb{R}$ and $\tilde\alpha \in (0,1)$ is such that $\phi_m, \phi \in C^{0,\tilde\alpha}_{\tilde\nu}$ with $\phi_m \in Y^{\tilde\alpha}_{\tilde\nu}$ (recall this is the subspace of functions with no $k = 0, 1$ modes in $C^{0,\tilde\alpha}_{\tilde\nu}$) and $\phi_m \to \phi$ in $C_{loc}(\overline{B_1}\setminus\{0\})$. Then $\phi \in Y^{\tilde\alpha}_{\tilde\nu}$.
**Case I.** Let $0 < \beta < \alpha$ (but close) and suppose we have $\hat\nu < \nu$ but close. Then, see [34], we have $C^{2,\alpha}_{\nu,D} \subset\subset C^{2,\beta}_{\hat\nu,D}$; and so we have $X^\alpha_\nu \subset\subset X^\beta_{\hat\nu}$. From this we see there is some $\phi \in X^\beta_{\hat\nu}$ such that (after passing to a subsequence) $\phi_m \to \phi$ in $X^\beta_{\hat\nu}$. Note that we can pass to the limit in the equation for $\phi_m$ to see that $\phi$ satisfies $L_\gamma(\phi) = 0$ in $B\setminus\{0\}$ with $\phi = 0$ on $\partial B$; here $\gamma_m \to \gamma$.

Checking the details of the proof of Lemma 2 one sees that for $\beta$, $\hat\nu$ as above and close enough to $\alpha, \nu$ we can apply the result of the lemma for the new parameters to see that $\phi = 0$. From this we see that $\phi_m \to 0$ in $C^{2,\beta}_{\hat\nu,D}$. We now again consider the equation for $\phi_m$ and note that

$$-\Delta\phi_m = f_m - \gamma_m p\tau^{p-1}C_p^{p-1}\frac{x}{|x|^2}\cdot\nabla\phi_m \quad B_1\setminus\{0\}$$

with $\phi_m = 0$ on $\partial B_1$.

**Claim 1.** *We claim that for $\hat{\nu}$ and $\beta$ chosen as above (but close enough) we have that*

$$\gamma_m p\tau^{p-1} C_p^{p-1} \frac{x}{|x|^2} \cdot \nabla\phi_m \to 0 \quad \text{in } Y^{\alpha}_{\hat{\nu}-2}.$$

*Assuming this claim for now; we can then apply the linear theory for $-\Delta$ to see that $\phi_m \to 0$ in $C^{2,\alpha}_{\hat{\nu},D}$. In particular we have $\phi_m \to 0$ in $C^{2,\alpha}(\overline{B_1 \backslash B_{\frac{1}{4}}})$; a contradiction.*

**Proof of Claim 1.** First note that since $\phi_m \to 0$ in $C^{2,\beta}_{\hat{\nu},D}$ and so $g_m := \frac{x}{|x|^2} \cdot \nabla\phi_m \to 0$ in $C^{0,\alpha}(B_1 \backslash B_{\frac{1}{4}})$. Also note that for $0 < s_m < \frac{1}{4}$ we have

$$s_m^{2-\hat{\nu}} \sup_{A_{s_m}} |g_m| \le C s^{1-\hat{\nu}} \sup_{A_{s_m}} |\nabla\phi_m| \le C_1 \|\phi_m\|_{C^{2,\beta}_{\hat{\nu},D}} \to 0.$$

We now consider the final term in the norm. First we note a scaling argument shows there is some $C > 0$

$$\sup_{x,y \in A_{s_m}} \frac{|\nabla\phi_m(x) - \nabla\phi_m(y)|}{|x-y|} \le C s_m^{\hat{\nu}-2} \|\phi_m\|_{C^{2,\beta}_{\hat{\nu}}}.$$

Let $0 < s_m < \frac{1}{4}$ and $x, y \in A_{s_m}$. Then we have

$$|g_m(x) - g_m(y)| \le \frac{|\nabla\phi_m(x) - \nabla\phi_m(y)|}{|x|} + \frac{|\nabla\phi_m(y)||x-y|}{|x|^2} + \frac{|\nabla\phi_m(y)|(|y|+|x|)|y-x|}{|x|^2|y|},$$

and using this we see

$$s_m^{2-\hat{\nu}+\alpha} \frac{|g_m(x) - g_m(y)|}{|x-y|^{\alpha}} \le \frac{C s_m^{2-\hat{\nu}+\alpha}}{s_m} \frac{|\nabla\phi_m(x) - \nabla\phi_m(y)|}{|x-y|^{\alpha}} + \frac{C s_m^{2-\hat{\nu}+\alpha} |\nabla\phi_m(y)|x-y|^{1-\alpha}}{s_m^2} + \frac{C s_m^{2-\hat{\nu}+\alpha} |\nabla\phi_m(y)||x-y|^{1-\alpha}}{s_m^2}$$

and from this we obtain some $C_0 > 0$ such that

$$s_m^{2-\hat{\nu}+\alpha} \frac{|g_m(x) - g_m(y)|}{|x-y|^{\alpha}} \le C_1 \|\phi_m\|_{C^{2,\beta}_{\hat{\nu}}} \to 0,$$

which completes the proof of the Claim 1. □

**Case II.** There is some $\varepsilon_0 > 0$ and $0 < C \le \frac{1}{4}$ and some $s_m \in [C, \frac{1}{4}]$

$$s_m^{-\nu} \left\{ \sup_{A_{s_m}} |\phi_m| + s_m \sup_{A_{s_m}} |\nabla\phi_m| + s_m^2 \sup_{A_{s_m}} |D^2\phi_m| + s_m^{2+\alpha} \sup_{x,y \in A_{s_m}} \frac{|D^2\phi_m(x) - D^2\phi_m(y)|}{|x-y|^{\alpha}} \right\} \ge \varepsilon_0. \tag{11}$$

We argue exactly as in Case I to see that $\phi_m \to 0$ in $C^{2,\alpha}_{\hat{\nu},D}$. But since $s_m$ is bounded away from zero we see the weight $\nu$ does not play a role in the quantity (11). From this we see that this quantity in (11) goes to zero as $m \to \infty$; a contradiction.

**Case III.** We now consider four subcases. We assume there is some $\varepsilon_0 > 0$, $s_m \to 0$ and $x_m \in A_{s_m}$ such that one of the following holds:

**Case 1.** $s_m^{-\nu}|\phi_m(x_m)| \ge \varepsilon_0$,
**Case 2.** $s_m^{1-\nu}|\nabla\phi_m(x_m)| \ge \varepsilon_0$,
**Case 3.** $s_m^{2-\nu}|D^2\phi_m(x_m)| \ge \varepsilon_0$.
**Case 4.** some $y_m \in A_{s_m}$ such that $s_m^{2-\nu+\alpha}\frac{|D^2\phi_m(x_m)-D^2\phi_m(y_m)|}{|x_m-y_m|^\alpha} \ge \varepsilon_0$.

We now consider each of these four cases. Note in all these cases we have the supremum over $A_{s_m}$ is bounded above.

**Case 1.** Define

$$\psi_m(x) := s_m^{-\nu}\phi_m(s_m x), \qquad |x| < \frac{1}{s_m},$$

and note there is some $1 \le |z_m| \le 2$ such that $|\psi_m(z_m)| \ge \varepsilon_0$. Set $E_k := \{x \in \mathbb{R}^N : \frac{1}{k} < |x| < k\}$ for $k \ge 2$ an integer. Note that $\psi_m$ satisfies

$$L_m(\psi_m) = -\Delta\psi_m(x) + \frac{\gamma_m p\tau^{p-1}C_p^{p-1}x\cdot\nabla\psi_m(x)}{|x|^2} = s_m^{2-\nu}f_m(s_m x) \qquad |x| < \frac{1}{s_m}. \tag{12}$$

Note that on $E_k$ we have the right hand side of this equation converge uniformly to zero. Using the assumption that $\|\phi_m\|_{C^{2,\alpha}_\nu} = 1$ we get

$$\sup_{0<s<\frac{1}{4}} s^{-\nu}\sup_{A_s}|\phi_m| \le 1,$$

that gives

$$s^{-\nu}|\phi_m(x)| \le 1, \ \ 0 < s < \frac{1}{4}, \ s < |x| < 2s.$$

Changing the variables $x \to tx$ and $s \to \delta t$ in the above we get

$$t^{-\nu}|\phi_m(tx)| \le \delta^\nu, \ \ 0 < \delta < \frac{1}{4t}, \ \delta < |x| < 2\delta.$$

Now taking $t = s_m$ in the above inequality and using $\delta^\nu < (\frac{|x|}{2})^\nu$ we get, for all $0 < \delta < \frac{1}{4s_m}$

$$|\psi_m(x)| \le \frac{|x|^\nu}{2^\nu} \qquad \text{for } \delta < |x| < 2\delta.$$

In particular we have this bound on $E_k$ for each fixed $k$ for large enough $m$. Note also that for each fixed $k$ and all large $m$ with $s_m < \frac{1}{k}$, $\psi_m$ is uniformly bounded in $C^{2,\alpha}(E_k)$. To see this note that from the definition of $\psi_m$ we have

$$\|\nabla \psi_m\|_{L^\infty(E_k)} = s_m^{1-\nu} \sup_{x\in E_k} |\nabla \phi_m(s_m x)| = s_m^{1-\nu} \sup_{y\in s_m E_k} |\nabla \phi_m(y)|.$$

Now assume $\sup_{y\in s_m E_k} |\nabla \phi_m(y)| = \sup_{y\in A_{s'_m}} |\nabla \phi_m(y)|$ for a subset $A_{s'_m} \subseteq s_m E_k$ then (note we must have $\frac{s_m}{s'_m} < k$) we have

$$s_m^{1-\nu} \sup_{y\in s_m E_k} |\nabla \phi_m(y)| = (\frac{s_m}{s'_m})^{1-\nu} (s'_m)^{1-\nu} \sup_{y\in A_{s'_m}} |\nabla \phi_m(y)| \le k^{1-\nu} \|\phi_m\|_{C^{2,\alpha}_\nu} = k^{1-\nu},$$

where in the last inequality we used the assumption that $1 = \|\phi_m\|_{C^{2,\alpha}_\nu}$. Similarly we can obtain $\|D^2\psi_m\|_{L^\infty(E_k)} \le k^{2-\nu}$ and also find a uniform bound for the Hölder norm of $D^2\psi_m$ on $E_k$. Now since $E_k \subseteq E_{k+1}$ and $\bigcup_1^\infty E_k = \mathbb{R}^N \setminus \{0\}$ then by the above estimates $\psi_m$ is locally uniformly bounded in $C^{2,\alpha}(\mathbb{R}^N \setminus \{0\})$. Hence by the Arzela–Ascoli theorem and a standard diagonal argument, up to a subsequence, $\psi_m$ converges at least in $C^{1,\alpha}_{loc}(\mathbb{R}^N \setminus \{0\})$ to a function $\psi$ which is a solution of

$$L_{\gamma_\infty}(\psi) = -\Delta\psi(x) + \frac{\gamma_\infty p\tau^{p-1} C_p^{p-1} x\cdot\nabla\psi(x)}{|x|^2} = 0 \ \text{ in } \ \mathbb{R}^N\backslash\{0\}, \tag{13}$$

with $\lim_{|x|\to\infty} |\psi(x)| = 0$ and $|\psi(x)| \le 2^{-\nu}|x|^\nu$. Here we have $\gamma_m \to \gamma_\infty \in [0,1]$. Note also that since each $\phi_m$ had no $k = 0, 1$ modes we see that $\psi$ has no $k = 0, 1$ modes. Then by Lemma 3 we have $\psi = 0$, but recalling $\psi_m \to \psi = 0$ in $C^{1,\alpha}(E_k)$ for each $k \ge 2$ we see that $\psi_m(z_m) \to 0$; a contradiction.

**Case 2.** Define $\psi_m(x) := s_m^{-\nu}\phi_m(s_m x)$. From our assumption on $\phi_m$ there is some $|z_m| \in (1,2)$ such that $|\nabla\psi_m(z_m)| \ge \varepsilon_0$. Now note that $\psi_m$ solves (12) and we can pass to the limit as in case 1. So as in case 1 we have $\psi_m \to \psi = 0$ in $C^{1,\gamma}(E_k)$ for each $k \ge 2$. In particular we have $|\nabla\psi_m(z_m)| \to 0$; a contradiction.

**Case 3.** Again we set $\psi_m(x) := s_m^{-\nu}\phi_m(s_m x)$. We now suppose the result does not hold and so there is some $\varepsilon_0 > 0$ and some $1 < |z_m| < 2$ such that $|D^2\psi_m(z_m)| \ge \varepsilon_0$. As before we have $\psi_m$ satisfies (12) on $E_k$. Set $g_m(x) := s_m^{2-\nu} f_m(s_m x)$ denote the right hand side of (12). As before we have $g_m \to 0$ in $L^\infty(E_k)$ for each fixed $k$. We now check the Hölder portion of the norm. Note for $x, y \in E_k$ ($k$ fixed and $m$ large) we have

$$\frac{|g_m(x) - g_m(y)|}{|x-y|^\alpha} = s_m^{2-\nu+\alpha} \frac{|f_m(s_m x) - f_m(s_m y)|}{|s_m x - s_m y|^\alpha}.$$

Then note we have the existence of some $C_k$ such that

$$\frac{|g_m(x) - g_m(y)|}{|x-y|^\alpha} = s_m^{2-\nu+\alpha} \frac{|f_m(s_m x) - f_m(s_m y)|}{|s_m x - s_m y|^\alpha} \le C_k \|f_m\|_{C^{0,\alpha}_{\nu-2}}$$

and hence we have $g_m \to 0$ in $C^{0,\alpha}(E_k)$ for all $2 \le k$. We now claim that $\psi_m \to 0$ in $C^{2,\alpha}(E_k)$ for all $k \ge 2$ and in particular we have the result for $k = 2$; which gives

$$s_m^{2-\nu} \sup_{\frac{s_m}{2} < |x| < 2s_m} |D^2\phi_m(x)| = \sup_{x\in E_2} |D^2\psi_m(x)| \to 0, \quad \text{and}$$

$$s_m^{2-\nu+\alpha} \sup_{x,y\in E_2} \frac{|D^2\phi_m(s_m x) - D^2\phi_m(s_m y)|}{|s_m x - s_m y|^\alpha} = \sup_{x,y\in E_2} \frac{|D^2\psi_m(x) - D^2\psi_m(y)|}{|x-y|^\alpha} \to 0,$$

which rules out case 3 and case 4.

We now prove the needed claim. We first fix $0 < \beta < \alpha$ but close and note by a compactness argument (and a diagonal argument) there is some $\psi$ such that $\psi_m \to \psi$ in $C^{2,\beta}(E_k)$ for all $k \geq 2$. So as above $\psi$ satisfies the limiting equation and we can use the previous results to see that $\psi = 0$ in $E_k$. We now rewrite (12) as

$$-\Delta\psi_m(x) = s_m^{2-\nu} f_m(s_m x) - \frac{\gamma_m p\tau^{p-1}C_p^{p-1} x\cdot\nabla\psi_m(x)}{|x|^2} \qquad |x| < \frac{1}{s_m}, \tag{14}$$

and note that since $\psi_m \to 0$ in $C^{2,\beta}(E_{2k})$ we must have $\frac{\gamma_m p\tau^{p-1}C_p^{p-1} x\cdot\nabla\psi_m(x)}{|x|^2} \to 0$ in $C^{0,\alpha}(E_{2k})$ and hence we can use interior estimates to obtain the desired convergence to result on $E_k$. □

We now examine the linear operator $L$ on modes $k = 0$ and $k = 1$.

**Lemma 4** *($k = 0, 1$ modes). Let $N, p, \tau, \nu$ be as in (8). There is some $C > 0$ such that for all functions $b_0, b_1$ (with finite norm defined below) defined on $(0, 1]$ there is some $a_k$ defined on $(0, 1]$ with $a_k(1) = 0$ and which satisfies*

$$a_k''(r) + \frac{(N-1-p(N-2-\tau))}{r} a_k'(r) - \frac{\lambda_k}{r^2} a_k(r) = b_k(r), \quad 0 < r < 1.$$

*Moreover one has the estimate $\|a_k\|_{C^{2,\alpha}_\nu} \leq C\|b_k\|_{C^{0,\alpha}_{\nu-2}}$.*

**Proof.** As before we set $\beta := (N-1-p(N-2-\tau))$ and $b := \beta - 1$.
**Mode $k = 0$.** Using the integrating factor technique we have

$$\frac{d}{dr}(r^\beta a_0'(r)) = r^\beta b_0(r) \qquad 0 < r < 1,$$

and integrating this from $r$ to 1 and taking $a_0'(1) = 0$ as a free parameter we get

$$r^\beta a_0'(r) = -\int_r^1 t^\beta b_0(t)\,dt,$$

and using the bound on $b_0$ we see

$$|a_0'(r)| \leq \frac{\|b_0\|_{C^{0,\alpha}_{\nu-2}}}{r^\beta} \int_r^1 t^{\beta-2+\nu}\,dt.$$

Checking the parameter ranges one sees that $\beta - 2 + \nu < -1$ and hence there is some $C > 0$ such that

$$|a_0'(r)| \le C \frac{\|b_0\|_{C^{0,\alpha}_{\nu-2}}}{r^{1-\nu}},$$

which is the desired weighted $L^\infty$ estimate on $a_0'$. Integrating this estimate gives $r^{-\nu}|a_0(r)| \le C\|b\|_{C^{0,\alpha}_{\nu-2}}$. To obtain estimates for the other terms in the $C^{2,\alpha}_\nu$ norm one can work directly with the ode and combine the results to obtain an estimate of the form $\|a_0\|_{C^{2,\alpha}_\nu} \le C\|b_0\|_{C^{0,\alpha}_{\nu-2}}$.

**Mode $k = 1$.**

$$r^2 a_1''(r) + r(N-1-p(N-2-\tau))a_1'(r) - (N-1)a_1(r) = r^2 b_1(r) \quad 0 < r < 1$$

with $a_1(1) = 0$. (Note we have used fact that $\lambda_1 = N-1$.)

*Homogeneous solutions.* Try $a(r) = r^\alpha$ and we then get

$$\alpha^2 + b\alpha - (N-1) = 0$$

which has roots

$$\alpha_+ := \frac{-b}{2} + \frac{\sqrt{b^2+4(N-1)}}{2}$$
$$\alpha_- := \frac{-b}{2} - \frac{\sqrt{b^2+4(N-1)}}{2}.$$

Set $y_1(r) := r^{\alpha_+}, \quad y_2(r) := r^{\alpha_-}$ (we switched notation from $a$ to $y$ to allow us to index the solutions without confusion with regards to the modes). Variation of parameters says that

$$W(y_1, y_2)(r) = (\alpha_- + \alpha_+)r^{\alpha_+ + \alpha_- - 1},$$

and a particular solution would be given by

$$a_p(r) = u(r)y_1(r) + v(r)y_2(r)$$

where

$$u'(r) = \frac{-y_2(r)b_1(r)}{W(r)} = \frac{-b_1(r)}{(\alpha_+ + \alpha_-)r^{\alpha_+ - 1}}, \qquad v'(r) = \frac{y_1(r)b_1(r)}{w(r)} = \frac{b_1(r)}{(\alpha_+ + \alpha_-)r^{\alpha_- - 1}}.$$

Let $\gamma := \alpha_+ + \alpha_-$ and then note we can write the general solution in the form

$$a(r) = \frac{r^{\alpha_-}}{\gamma} \int_{T_1}^{r} \frac{b_1(t)}{t^{\alpha_- - 1}} dt - \frac{r^{\alpha_+}}{\gamma} \int_{1}^{r} \frac{b_1(t)}{t^{\alpha_+ - 1}} dt + C_1 r^{\alpha_+} + C_2 r^{\alpha_-},$$

where $T_1$ and $C_i$ are to be picked later.

The first thing to note that is that

$$\frac{|b_1(t)|}{t^{\alpha_- -1}} \le \frac{\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{t^{\alpha_- +\tau+1}}$$
$$\frac{|b_1(t)|}{t^{\alpha_+ -1}} \le \frac{\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{t^{\alpha_+ +\tau+1}}.$$

Since $\alpha_+ + \tau + 1 > 1$ we see that

$$\Big| \int_1^r \frac{b_1(t)}{t^{\alpha_+ -1}} dt \Big| \le \frac{C\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{r^{\alpha_+ +\tau}}$$

and hence

$$\Big| \frac{r^{\alpha_+}}{\gamma} \int_1^r \frac{b_1(t)}{t^{\alpha_+ -1}} dt \Big| \le \frac{C\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{r^{\tau}}.$$

With this in mind let us take $C_2 = 0$. So we now have

$$a(r) = \frac{r^{\alpha_-}}{\gamma} \int_{T_1}^r \frac{b_1(t)}{t^{\alpha_- -1}} dt - \frac{r^{\alpha_+}}{\gamma} \int_1^r \frac{b_1(t)}{t^{\alpha_+ -1}} dt + C_1 r^{\alpha_+}.$$

A computation shows that $\alpha_- + \tau < 0$. Indeed, this is equivalent to $b + \sqrt{b^2 + 8N} > 2\tau$ or $\tau - \frac{2N}{\tau} \le b = N - 2 - p(N - 2 - \tau)$. A calculation shows that the latter inequality is equivalent to $(p-1)(\tau+2) \ge N(p-1-\frac{2}{\tau}) = N(p-1)\frac{-p}{2-p}$, which is true. Hence, $\frac{b_1(t)}{t^{\alpha_- -1}}$ is integrable on $(0, 1)$ and so we can take $T_1 = 0$. So we have

$$a(r) = \frac{r^{\alpha_-}}{\gamma} \int_0^r \frac{b_1(t)}{t^{\alpha_- -1}} dt - \frac{r^{\alpha_+}}{\gamma} \int_1^r \frac{b_1(t)}{t^{\alpha_+ -1}} dt + C_1 r^{\alpha_+},$$

and we pick $C_1$ such that

$$0 = \frac{1}{\gamma} \int_0^1 \frac{b_1(t)}{t^{\alpha_- -1}} dt + C_1,$$

and hence $a(1) = 0$. Using the bound on $b_1$ we see that $|C_1| \le C\|b_1\|_{C^{0,\alpha}_{\nu-2}}$. We now get some estimates. First note that

$$\int_r^1 \frac{|b_1(t)|}{t^{\alpha_+ -1}} dt \le \|b_1\|_{C^{0,\alpha}_{\nu-2}} \int_r^1 \frac{1}{t^{1-\nu+\alpha_+}} dt,$$

and note that $1-\nu+\alpha_+ > 1$ and hence the right hand side is bounded above by $\frac{C\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{r^{\alpha_+-\nu}}$ and hence we have

$$\Big|\frac{r^{\alpha_+}}{\gamma}\int_1^r \frac{b_1(t)}{t^{\alpha_+-1}}dt\Big| \le \frac{C\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{r^{-\nu}}.$$

We now examine the other two terms and towards this define

$$z(r) := \frac{r^{\alpha_-}}{\gamma}\int_0^r \frac{b_1(t)}{t^{\alpha_--1}}dt + C_1 r^{\alpha_+}.$$

Using the bound on $b_1$ and $C_1$ we see that

$$|z(r)| \le \frac{C\|b_1\|_{C^{0,\alpha}_{\nu-2}}}{r^{-\nu}} + C\|b_1\|_{C^{0,\alpha}_{\nu-2}} r^{\alpha_+},$$

and hence $r^{-\nu}|z(r)| \le C\|b_1\|_{C^{0,\alpha}_{\nu-2}}(1+r^{\alpha_+-\nu})$ which gives us the desired weighted $L^\infty$ bound on $a(r)$. To get the desired weighted $L^\infty$ bounds on $a'(r)$ and $a''(r)$, differentiate the formula of $a(r)$ to get $a'(r)$ and do similar as above we, then we use the ODE for $a''(r)$. Combining them we get then the estimate $\|a\|_{C^{2,\alpha}_\nu} \le C\|b_1\|_{C^{0,\alpha}_{\nu-2}}$. $\square$

**Lemma 5** *(Combining the estimates). Let $N, p, \tau, \nu$ be as in (8). There is some $C > 0$ such that for all $f \in C^{0,\alpha}_{\nu-2}(\overline{B_1}\backslash\{0\})$ there is some $\phi \in C^{2,\alpha}_{\nu,D}(\overline{B_1}\backslash\{0\})$ such that $L(\phi) = f$ in $B_1\backslash\{0\}$. Moreover one has $\|\phi\|_{C^{2,\alpha}_{\nu,D}} \le C\|f\|_{C^{0,\alpha}_{\nu-2}}$.*

**Proof.** For this proof we are more precise with our notation regarding the eigenfunctions of the Laplace–Beltrami operator. First we have $\Theta_0(\theta) = 1$ and then for the $k = 1$ mode there is $\{\Theta_{1,i}(\theta) : 1 \le i \le N\}$ and then there is the higher modes. Given $f \in C^{0,\alpha}_{\nu-2}$ we write

$$f(x) = b_0(r) + \sum_{i=1}^N b_{1,i}(r)\Theta_{1,i}(\theta) + \hat{f}(x),$$

where $\hat{f} \in Y^\alpha_{\nu-2}$. For $\phi \in C^{2,\alpha}_{\nu,D}$ we similarly write

$$\phi(x) = a_0(r) + \sum_{i=1}^N a_{1,i}(r)\Theta_{1,i}(\theta) + \hat{\phi}(x), \tag{15}$$

where $\hat{\phi} \in X^\alpha_\nu$. We now get the desired estimates. Let $f \in C^{0,\alpha}_{\nu-2}$ with the above representation and we let $a_0, a_{1,i}, \hat{\phi}$ be such that $L(a_0) = b_0$, $L(a_{1,i}) = b_{1,i}$ for $1 \le i \le N$ and $L(\hat{\phi}) = \hat{f}$ where $\hat{\phi} \in X^\alpha_\nu$ and where all functions satisfy a zero Dirichelt boundary condition. By our earlier results there is some $C > 0$ such that

$$\|a_0\|_{C^{2,\alpha}_\nu} \le C\|b_0\|_{C^{0,\alpha}_{\nu-2}}, \quad \|a_{1,i}\|_{C^{2,\alpha}_\nu} \le C\|b_{1,i}\|_{C^{0,\alpha}_{\nu-2}}\ 1\le i\le N, \quad \|\hat{\phi}\|_{C^{2,\alpha}_\nu} \le C\|\hat{f}\|_{C^{0,\alpha}_{\nu-2}}.$$

We now define $\phi$ as in (15) and hence $L(\phi)=f$ in $B_1\backslash\{0\}$ with $\phi=0$ on $\partial B_1$. Additionally we have

$$\|\phi\|_{C^{2,\alpha}_\nu} \le C\left(\|b_0\|_{C^{0,\alpha}_{\nu-2}} + \sum_{i=1}^N \|b_{1,i}\|_{C^{0,\alpha}_{\nu-2}} + \|\hat{f}\|_{C^{0,\alpha}_{\nu-2}}\right),$$

and we now claim there is some $C_1>0$ such that

$$\left(\|b_0\|_{C^{0,\alpha}_{\nu-2}} + \sum_{i=1}^N \|b_{1,i}\|_{C^{0,\alpha}_{\nu-2}} + \|\hat{f}\|_{C^{0,\alpha}_{\nu-2}}\right) \le C_1\|b_0+\sum_{i=1}^N b_{1,i}\Theta_{1,i}+\hat{f}\|_{C^{0,\alpha}_{\nu-2}} = C_1\|f\|_{C^{0,\alpha}_{\nu-2}},$$

which would give our desired estimate. We now suppose the claim is false and so for all $m\ge 1$ there is some $b_0^m, b_{1,i}^m, \hat{f}^m$ such that

$$\|b_0^m\|_{C^{0,\alpha}_{\nu-2}} + \sum_{i=1}^N \|b_{1,i}^m\|_{C^{0,\alpha}_{\nu-2}} + \|\hat{f}^m\|_{C^{0,\alpha}_{\nu-2}} \ge m\|b_0^m + \sum_{i=1}^N b_{1,i}^m\Theta_{1,i} + \hat{f}^m\|_{C^{0,\alpha}_{\nu-2}}. \tag{16}$$

We now define $t_0^m := \|b_0^m\|_{C^{0,\alpha}_{\nu-2}}$ and $t_i^m := \|b_{1,i}^m\|_{C^{0,\alpha}_{\nu-2}}$ for $1\le i\le N$ and $t_{N+1}^m := \|\hat{f}^m\|_{C^{0,\alpha}_{\nu-2}}$. After passing to a subsequence in $m$ we can assume that there is some $0\le i_0\le N+1$ such that $t_{i_0}^m \ge t_i^m$ for all $0\le i\le N+1$.

We now re-normalize each term by dividing by $t_{i_0}^m$; we define (without using new notation) $b_0^m := \frac{b_0^m}{t_{i_0}^m}$, $b_{1,i}^m := \frac{b_{1,i}^m}{t_{i_0}^m}$, $\hat{f}^m := \frac{\hat{f}^m}{t_{i_0}^m}$. Note we still have (16) with these re-normalized functions, and note the left hand side of (16) is bounded below by 1 and above by some constant $C$. Note by (16) we have $f^m := b_0^m + \sum_{i=1}^N b_{1,i}^m\Theta_{1,i} + \hat{f}^m \to 0$ in $C^{0,\alpha}_{\nu-2}$.

We now suppose that $0\le i_0\le N$ and we now consider

$$g_m(x) := \int_{\theta\in S^{N-1}} f^m(|x|\theta)\Theta_{1,i_0}(\theta)d\theta,$$

where for notational convenience we are defining $\Theta_{1,0}(\theta)=1$ the $k=0$ eigenfunction. We claim that since $f^m\to 0$ in $C^{0,\alpha}_{\nu-2}$ that $g_m\to 0$ in $C^{0,\alpha}_{\nu-2}$; we will prove this claim later. But now note that $g_m(x)=b_{1,i_0}^m(|x|)$ where we are abusing notation again; we are taking $b_{1,i_0}^m(r)=b_0^m(r)$ in the case of $i_0=0$. So we have $b_{1,i_0}^m(r)\to 0$ in $C^{0,\alpha}_{\nu-2}$; which contradicts the fact that this quantity has norm 1. So from this we must have $i_0=N+1$ and we also that $b_0^m, b_{1,i}^m\to 0$ in $C^{0,\alpha}_{\nu-2}$ for all $0\le i\le N$. So we must have $\|\hat{f}^m\|_{C^{0,\alpha}_{\nu-2}}=1$. But recall we have (after applying the triangle inequality)

$$C \ge m\|f^m\|_{C^{0,\alpha}_{\nu-2}} \ge m\left(\|\hat{f}^m\|_{C^{0,\alpha}_{\nu-2}} - \varepsilon_m\right)$$

where $\varepsilon_m$ is the sum of the norms of $b_0^m(r)$, $b_{1,i}^m(r)\Theta_{1,i}(\theta)$. So $\varepsilon_m \to 0$ and since $\|\hat{f}^m\|_{C^{0,\alpha}_{\nu-2}} = 1$ we get a contradiction.

We now prove the needed claim. Without loss of generality take $1 \le i_0 \le N$. Its clear that $g_m \to 0$ uniformly on $B_1 \backslash B_{\frac{1}{4}}$. Now suppose $0 < s < \frac{1}{4}$ and let $\theta \in S^{n-1}$ and $x \in A_s$. Then one sees that

$$s^{2-\nu}|g_m(x)| \le C_{i_0} \int\limits_{\theta \in S^{N-1}} |f^m(|x|\theta)|s^{2-\nu} d\theta \le C_{i_0} C \|f^m\|_{C^{0,\alpha}_{\nu-2}}$$

and hence we see

$$\sup_{0<s<\frac{1}{4}} s^{2-\nu} \sup_{x \in A_s} |g_m(x)| \to 0.$$

We now need to consider the Hölder portions of the norm. We first let $x, y \in B_1 \backslash B_{\frac{1}{4}}$ distinct. Then we have

$$\frac{|g_m(x) - g_m(y)|}{|x-y|^\alpha} \le \int\limits_{\theta \in S^{N-1}} \frac{|f^m(|x|\theta) - f^m(|y|\theta)|}{||x|\theta - |y|\theta|^\alpha} I_1 |\Theta_{1,i_0}(\theta)| d\theta,$$

where

$$I_1 = \frac{||x|\theta - |y|\theta|^\alpha}{|x-y|^\alpha}.$$

First note that

$$\frac{|f^m(|x|\theta) - f^m(|y|\theta)|}{||x|\theta - |y|\theta|^\alpha} \le \|f^m\|_{C^{0,\alpha}_{\nu-2}}.$$

Also note by the triangle inequality we have $I_1 \le 1$. Hence from the above we see that

$$\sup_{x,y \in B_1 \backslash B_{\frac{1}{4}}} \frac{|g_m(x) - g_m(y)|}{|x-y|^\alpha} \to 0.$$

We now assume $0 < s < \frac{1}{4}$ and $x, y \in A_s$ are distinct. Using the above computations we see that

$$s^{2-\nu+\alpha} \frac{|g_m(x) - g_m(y)|}{|x-y|^\alpha} \le \int\limits_{\theta \in S^{N-1}} \|f^m\|_{C^{0,\alpha}_{\nu-2}} I_1 |\Theta_{1,i_0}(\theta)| d\theta.$$

Combing the results we see that $g_m \to 0$ in $C^{0,\alpha}_{\nu-2}$. $\square$

## 2. The fixed point argument

Recall we have defined $J_t(\phi) = \psi$ where $\psi$ satisfies (7). To obtain a solution $\phi$ of (6) we will show that $J_t$ is a contraction on $B_r$ where $B_r$ is the closed ball of radius $r$ centered at the origin in $C^{2,\alpha}_{\nu,D}(B_1\backslash\{0\})$ where as before we assume that $N, p, \tau, \nu$ satisfy (8). Recall the generalized Binomial theorem says for $a > 0$ and $|b| < a$ we can write

$$(a+b)^{\frac{p}{2}} = \sum_{k=0}^{\infty} \gamma_k a^{\frac{p}{2}-k} b^k,$$

where $\gamma_0 = 1$ and $\gamma_1 = \frac{p}{2}$. We use this to rewrite (7) as

$$\begin{aligned} L(\psi) &= \sum_{k=1}^{\infty} \gamma_k t^k |\nabla w + \nabla\phi|^{p-2k} \left((A_1(x)(\nabla w(x) + \nabla\phi(x))) \cdot (\nabla w(x) + \nabla\phi(x))\right)^k \\ &\quad + |\nabla w + \nabla\phi|^p - |\nabla w|^p - p|\nabla w|^{p-2}\nabla w \cdot \nabla\phi \\ &\quad + E_t(w) + E_t(\phi) \\ &=: K_1(\phi) + K_2(\phi) + E_t(w) + E_t(\phi) \qquad \text{in } B_1\backslash\{0\} \end{aligned} \tag{17}$$

with $\psi = 0$ on $\partial B_1$.

We now begin with some computations. To simplify the calculus in the weighted Hölder spaces we use the following properties and remarks about these spaces.

**Lemma 6** *(Proposition 2.1 and Lemma 2.2 [34], Lemma 1 in [31]). The following properties hold.*

(i) *Assume that $u \in C^{k+1,\alpha}_{\nu}(B_1 \setminus \{0\})$ then $\nabla u \in C^{k,\alpha}_{\nu-1}(B_1 \setminus \{0\})$.*

(ii) *If $u_i \in C^{k,\alpha}_{\nu_i}(B_1 \setminus \{0\})$, $i = 1, 2$ then $u_1 u_2 \in C^{k,\alpha}_{\nu_1+\nu_2}(B_1 \setminus \{0\})$ and*

$$\|u_1 u_2\|_{C^{k,\alpha}_{\nu_1+\nu_2}(B_1\backslash\{0\})} \le c\|u_1\|_{C^{k,\alpha}_{\nu_1}} \|u_2\|_{C^{k,\alpha}_{\nu_2}(B_1\backslash\{0\})},$$

*for some constant $c > 0$ independent of $u_1$ and $u_2$.*

(iii) *If $0 < u \in C^{k,\alpha}_{\nu}(B_1 \setminus \{0\})$ and $q > 0$ then $u^q \in C^{k,\alpha}_{q\nu}(B_1 \setminus \{0\})$. In addition*

$$\|u^q\|_{C^{k,\alpha}_{q\nu}(B_1\backslash\{0\})} \le c\|u\|^q_{C^{k,\alpha}_{\nu}(B_1\backslash\{0\})}$$

*for some constant $c > 0$ which does not depend on $u$.*

(iii′) *Also, if we replace $u > 0$ in above with $u \ge 0$ then we have the same if $q > k + 1$.*

(iv) *If $k + \alpha < k' + \alpha'$ and $\nu < \nu'$ then the embedding $C^{k',\alpha'}_{\nu'} \hookrightarrow C^{k,\alpha}_{\nu}$ is compact.*

**Remark 1.** Note that in Lemma 6 part (iii), if $0 < q$ is an integer then we can allow $u$ to be zero somewhere or change sign. To see this, use part (ii) for $q$ functions $u_1 = u_2 = ... = u_q = u$ and $\nu_1 = \nu_2 = ... = \nu_q = \nu$.

**Computations involving $K_2$.** Recall we have

$$K_2 = |\nabla w + \nabla\phi|^p - |\nabla w|^p - p|\nabla w|^{p-2}\nabla w \cdot \nabla\phi,$$

where $w = C_p(|x|^{-\tau} - 1)$ and $\phi \in B_r \subset C^{2,\alpha}_{D,\nu}$.

We show that when $\nu = -\tau$ then

$$\|K_2\|_{C^{0,\alpha}_{\nu-2}} \le C\|\phi\|^2_{C^{2,\alpha}_{\nu}},$$

for $r$ sufficiently small. We have, by the binomial expansion

$$K_2 = \frac{p}{2}|\nabla w|^{p-2}|\nabla\phi|^2 + \sum_{k=2}^{\infty}\gamma_k|\nabla w|^{p-2k}(2\nabla w\cdot\nabla\phi + |\nabla\phi|^2)^k := \frac{p}{2}f_1 + f_2.$$

Note that we have convergence provided $|2\nabla w \cdot \nabla\phi + |\nabla\phi|^2| < |\nabla w|^2$; which will easily be satisfied provided we take $\phi$ small in $C^{2,\alpha}_{\nu}$. First we estimate $f_1$. Note we have $|\nabla w| = C|x|^{\nu-1}$ so

$$f_1(x) = |x|^{(2-p)(-\nu+1)}|\nabla\phi|^2.$$

Note we have $|x|^{(2-p)(-\nu+1)} \in C^{1,\alpha}_{(2-p)(-\nu+1)}$. Also, by part (i) of Lemma 6, $\nabla\phi \in C^{1,\alpha}_{\nu-1}$ and thus by part (ii), $|\nabla\phi|^2 = \nabla\phi\cdot\nabla\phi \in C^{1,\alpha}_{2(\nu-1)}$ and

$$\||\nabla\phi|^2\|_{C^{1,\alpha}_{2(\nu-1)}} \le c\|\nabla\phi\|^2_{C^{1,\alpha}_{\nu-1}}.$$

Since $(2-p)(-\nu+1)+2(\nu-1) = \nu-2$ then using part (ii) of Lemma 6 again, we get $f_1 \in C^{1,\alpha}_{\nu-2}$ with

$$\|f_1\|_{C^{1,\alpha}_{\nu-2}} \le C\|\nabla\phi\|^2_{C^{1,\alpha}_{\nu-1}}.$$

And since $C^{1,\alpha}_{\nu-1} \hookrightarrow C^{0,\alpha}_{\nu-2}$ and $\|\nabla\phi\|_{C^{1,\alpha}_{\nu-1}} \le \|\phi\|_{C^{2,\alpha}_{\nu}}$ we get

$$\|f_1\|_{C^{0,\alpha}_{\nu-2}} \le C\|\phi\|^2_{C^{2,\alpha}_{\nu}}, \tag{18}$$

where $C$ is independent of $\phi$. To estimate $\|f_2\|_{C^{0,\alpha}_{\nu-2}}$ we write

$$|\nabla w|^{p-2k}(2\nabla w\cdot\nabla\phi + |\nabla\phi|^2)^k = |\nabla w|^p\Big(\frac{2\nabla w\cdot\nabla\phi + |\nabla\phi|^2}{|\nabla w|^2}\Big)^k := |\nabla w|^p a(x)^k.$$

We have $2\nabla w\cdot\nabla\phi \in C^{1,\alpha}_{2(\nu-1)}$ and $|\nabla\phi|^2 \in C^{1,\alpha}_{2(\nu-1)}$, hence $2\nabla w\cdot\nabla\phi + |\nabla\phi|^2 \in C^{1,\alpha}_{2(\nu-1)}$. Also, $\frac{1}{|\nabla w|^2} = C|x|^{2(1-\nu)} \in C^{1,\alpha}_{2(1-\nu)}$, hence

$$a(x) = \frac{1}{|\nabla w|^2}(2\nabla w\cdot\nabla\phi + |\nabla\phi|^2) \in C^{1,\alpha}_0.$$

Also, by Lemma 6 and the triangle inequality

$$\|a\|_{C_0^{1,\alpha}} \le C_0\|\nabla\phi\|_{C_{\nu-1}^{1,\alpha}} \le C_0\|\phi\|_{C_\nu^{2,\alpha}}.$$

Since $|\nabla w|^p \in C^{1,\alpha}_{p(\nu-1)} = C^{1,\alpha}_{\nu-2}$, then using again part (ii) of Lemma 6 we get

$$|\nabla w|^p a(x)^k \in C^{1,\alpha}_{\nu-2} \hookrightarrow C^{0,\alpha}_{\nu-2}, \quad \text{for } k = 2, 3, ...,$$

and

$$\begin{aligned}\| \, |\nabla w|^p a^k\|_{C^{0,\alpha}_{\nu-2}} &\le CC_0^k\|\phi\|^k_{C^{2,\alpha}_\nu} = CC_0^2\Big(C_0\|\phi\|_{C^{2,\alpha}_\nu}\Big)^{k-2}\|\phi\|^2_{C^{2,\alpha}_\nu} \\ &= Cc(r)^{k-2}\|\phi\|^2_{C^{2,\alpha}_\nu}, \quad \text{for } k = 2, 3, ...,\end{aligned}$$

where $c(r) \to 0$ as $r \to 0$. Hence

$$\|f_2\|_{C^{0,\alpha}_{\nu-2}} \le C\Big(\sum_{k=2}^{\infty}|\gamma_k|(c(r))^{k-2}\Big)\|\nabla\phi\|^2_{C^{2,\alpha}_\nu} \le C'\|\nabla\phi\|^2_{C^{2,\alpha}_\nu}, \tag{19}$$

for $r$ small. Now using the above estimates (18) and (19) we get

$$\|K_2\|_{C^{0,\alpha}_{\nu-2}} \le C\|\phi\|^2_{C^{2,\alpha}_\nu}, \tag{20}$$

for all $\phi \in B_r \subset C^{2,\alpha}_{D,\nu}$ with sufficiently small $r$.

**Computations involving $K_1$.** Recall we have, with $v = w + \phi$,

$$K_1(x) = \sum_{k=1}^{\infty}\gamma_k t^k|\nabla v|^{p-2k}(A_1(x)\nabla v\cdot\nabla v)^k = |\nabla v|^p\sum_{k=1}^{\infty}\gamma_k t^k(\frac{A_1(x)\nabla v\cdot\nabla v}{|\nabla v|^2})^k.$$

Taking $r$ sufficiently small we have $|\nabla v|$ bounded away from zero. We have $\nabla v \in C^{1,\alpha}_{\nu-1}$ then we show that $\frac{1}{|\nabla\ v|^2} \in C^{1,\alpha}_{2-2\nu}$. We write

$$\frac{1}{|\nabla v|^2} = \frac{1}{|\nabla w|^2 + |\nabla\phi|^2 + 2\nabla w\cdot\nabla\phi} = \frac{1}{|\nabla w|^2}\,\frac{1}{1+\frac{|\nabla\phi|^2+2\nabla w\cdot\nabla\phi}{|\nabla w|^2}} = |\nabla w|^{-2}\frac{1}{1+a(x)}.$$

Note we have $|a(x)| < 1$ for $r$ sufficiently small, and as we did in the first part we have $a \in C_0^{1,\alpha}$ and

$$\|a\|_{C_0^{1,\alpha}} \le C_0\|\phi\|_{C^{2,\alpha}_\nu} := c(r) \to 0 \text{ as } r \to 0.$$

Also we have

$$\frac{1}{1+a(x)} = \sum_{i=0}^{\infty}(-1)^i a^i(x).$$

Note that by Lemma 6 part (iii′), $a^i \in C_0^{1,\alpha}$ for every $i = 0, 1, 2, \ldots$ and $\|a^i\|_{C_0^{1,\alpha}} \le \|a\|^i_{C_0^{1,\alpha}} \le c^i(r)$, and since $C_0^{1,\alpha}$ is a Banach space then

$$\frac{1}{1+a(x)} \in C_0^{1,\alpha} \quad \text{and} \quad \left\| \frac{1}{1+a} \right\|_{C_0^{1,\alpha}} \le \sum_{i=0}^{\infty} c^i(r) = \frac{1}{1-c(r)} < 2, \tag{21}$$

for $r$ small. Now note that we have $|\nabla w|^{-2} = C|x|^{2-2\nu}$ so by Lemma 6 and (21), $\frac{1}{|\nabla v|^2} = |\nabla w|^{-2}\frac{1}{1+a(x)} \in C^{1,\alpha}_{2-2\nu}$ and

$$\left\| \frac{1}{|\nabla v|^2} \right\|_{C^{1,\alpha}_{2-2\nu}} \le c||C|x|^{2-2\nu}||_{C^{1,\alpha}_{2-2\nu}} \left\| \frac{1}{1+a} \right\|_{C_0^{1,\alpha}} := C_0,$$

with $C_0$ independent of $r$ for all small $r$. Now since $A(x)$ is smooth (we need here only $A(x) \in C_0^{1,\alpha}$) and $\nabla v \in C^{1,\alpha}_{\nu-1}$ then we easily get $B(x) := A(x)\nabla v \cdot \nabla v \in C^{1,\alpha}_{2\nu-2}$ with

$$\|B\|_{C^{1,\alpha}_{2\nu-2}} \le C\|\nabla v\|^2_{C^{1,\alpha}_{\nu-1}} \le C\|v\|^2_{C^{2,\alpha}_{\nu}}.$$

Summing up the above we get by Lemma 6

$$\frac{A_1(x)\nabla v \cdot \nabla v}{|\nabla v|^2} \in C_0^{1,\alpha}, \quad \text{with} \quad \left\| \frac{A_1 \nabla v \cdot \nabla v}{|\nabla v|^2} \right\|_{C_0^{1,\alpha}} \le C\|v\|^2_{C^{2,\alpha}_{\nu}} \le C,$$

where $C$ is independent of $r$ for all small $r$. Now using $|\nabla v|^p \in C^{1,\alpha}_{p(\nu-1)} = C^{1,\alpha}_{\nu-2}$ (note we used Lemma 6 part (iii) with $|\nabla v| > 0$), $C^{1,\alpha}_{\nu-2} \hookrightarrow C^{0,\alpha}_{\nu-2}$ (continuously) and the above estimates we get $K_1 \in C^{1,\alpha}_{\nu-2} \subset C^{0,\alpha}_{\nu-2}$ and

$$\|K_1\|_{C^{0,\alpha}_{\nu-2}} \le \Big(\sum_{k=1}^{\infty} |\gamma_k|(Ct)^k\Big)\|v\|^p_{C^{2,\alpha}_{\nu}} \le C(t) \to 0 \ \text{ as } \ t \to 0. \tag{22}$$

**Computations involving $E_t(w)$ and $E_t(\phi)$.** To estimate $\|E_t(w)\|_{C^{0,\alpha}_{\nu-2}}$ and $\|E_t(\phi)\|_{C^{0,\alpha}_{\nu-2}}$ we can use again Lemma 6 to easily get

$$\|E_t(w)\|_{C^{0,\alpha}_{\nu-2}} \le Ct \ \text{ and } \ \|E_t(\phi)\|_{C^{0,\alpha}_{\nu-2}} \le Ct\|\phi\|_{C^{2,\alpha}_{\nu}}. \tag{23}$$

Now by the definition of $J_t$, the continuity of the right inverse of $L$ and the above estimates (20), (22) and (23) we get

$$\begin{aligned}\|J_t(\phi)\|_{C^{2,\alpha}_{\nu}} &\le C\Big(\|K_1\|_{C^{0,\alpha}_{\nu-2}} + \|K_2\|_{C^{0,\alpha}_{\nu-2}} + \|E_t(w)\|_{C^{0,\alpha}_{\nu-2}} + \|E_t(\phi)\|_{C^{0,\alpha}_{\nu-2}}\Big)\\ &\le C(t) + C\|\phi\|^2_{C^{2,\alpha}_{\nu}} + Ct + Ct\|\phi\|_{C^{2,\alpha}_{\nu}},\end{aligned}$$

where $C(t) \to 0$ as $t \to 0$.

**Contraction.** Let $J_t(\phi) = \psi$ and $J_t(\phi_0) = \psi_0$ with $\phi, \phi_0 \in B_r$. Then we have

$$L(\psi - \psi_0) = E_t(\phi - \phi_0) + K_1(\phi) - K_1(\phi_0) + K_2(\phi) - K_2(\phi_0).$$

First we estimate $\|K_2(\phi) - K_2(\phi_0)\|_{C^{0,\alpha}_{\nu-2}}$. We have

$$\begin{aligned} K_2(\phi) - K_2(\phi_0) &= \frac{p}{2}|\nabla w|^{p-2}(|\nabla\phi|^2 - |\nabla\phi_0|^2) \\ &\quad + \sum_{k=2}^{\infty} \gamma_k |\nabla w|^p \left\{ \left(\frac{2\nabla w\cdot\nabla\phi + |\nabla\phi|^2}{|\nabla w|^2}\right)^k - \left(\frac{2\nabla w\cdot\nabla\phi_0 + |\nabla\phi_0|^2}{|\nabla w|^2}\right)^k \right\} \\ &:= F_1(x) + F_2(x). \end{aligned}$$

First note that as we showed before $F_1 \in \mathcal{C}^{1,\alpha}_{\nu-2} \hookrightarrow \mathcal{C}^{0,\alpha}_{\nu-2}$ (continuously), also we can write

$$F_1(x) = \frac{p}{2}|\nabla w|^{p-2}(\nabla\phi + \nabla\phi_0)\cdot\nabla(\phi - \phi_0),$$

hence

$$\|F_1\|_{C^{0,\alpha}_{\nu-2}} \le C(r)\|\phi - \phi_0\|_{C^{2,\alpha}_{\nu}}, \quad C(r) \to 0 \text{ as } r \to 0. \tag{24}$$

Also using the formula $a^k - b^k = (a-b)\sum_{i=0}^{k-1} a^i b^{k-1-i}$ for $k \ge 2$, we can write

$$\begin{aligned} a_1(x) &:= \left(\frac{2\nabla w\cdot\nabla\phi + |\nabla\phi|^2}{|\nabla w|^2}\right)^k - \left(\frac{2\nabla w\cdot\nabla\phi_0 + |\nabla\phi_0|^2}{|\nabla w|^2}\right)^k \\ &= \frac{(2\nabla w + \nabla\phi + \nabla\phi_0)\cdot\nabla(\phi - \phi_0)}{|\nabla w|^2} \\ &\quad \times \sum_{i=0}^{k-1}\left(\frac{2\nabla w\cdot\nabla\phi + |\nabla\phi|^2}{|\nabla w|^2}\right)^i \left(\frac{2\nabla w\cdot\nabla\phi_0 + |\nabla\phi_0|^2}{|\nabla w|^2}\right)^{k-1-i} \\ &:= a_2(x)\sum_{i=0}^{k-1} a_3(x)^i a_4(x)^{k-1-i}. \end{aligned}$$

And similar as we have done in the first part we have $a_j(x) \in C^{1,\alpha}_0$, $j = 1, ..., 4$ and

$$\|a_2\|_{C^{1,\alpha}_0} \le C\|\phi - \phi_0\|_{C^{2,\alpha}_{\nu}}, \quad \text{and} \quad \|a_j\|_{C^{1,\alpha}_0} \le c(r), \quad j = 3, 4, \quad c(r) \to 0 \text{ as } r \to 0.$$

Thus, using Lemma 6,

$$\|a_1\|_{C^{1,\alpha}_0} \le Ckc(r)^{k-1}\|\phi - \phi_0\|_{C^{2,\alpha}_{\nu}},$$

that gives

$$\|F_2\|_{C^{0,\alpha}_{\nu-2}} \le C\Big(\sum_{k=2}^{\infty}|\gamma_k|kc(r)^{k-1}\Big)\|\phi-\phi_0\|_{C^{2,\alpha}_{\nu}} = Cc(r)\Big(\sum_{k=2}^{\infty}|\gamma_k|kc(r)^{k-2}\Big)\|\phi-\phi_0\|_{C^{2,\alpha}_{\nu}},$$
$$c(r)\to 0 \text{ as } r\to 0. \tag{25}$$

And since $r$ small we have $kc(r)^{k-2} \le (\frac{1}{2})^k$, $k=3,4,\ldots$ then from (25) we get

$$\|F_2\|_{C^{0,\alpha}_{\nu-2}} \le c(r)\|\phi-\phi_0\|_{C^{2,\alpha}_{\nu}}, \quad c(r)\to 0 \text{ as } r\to 0. \tag{26}$$

Combining (24) and (26) we arrive at

$$\|K_2(\phi)-K_2(\phi_0)\|_{C^{0,\alpha}_{\nu-2}} \le C(r)\|\phi-\phi_0\|_{C^{2,\alpha}_{\nu}}, \quad C(r)\to 0 \text{ as } r\to 0. \tag{27}$$

Now we estimate $\|K_1(\phi)-K_1(\phi_0)\|_{C^{0,\alpha}_{\nu-2}}$. Taking $v=w+\phi$ and $v_0=w+\phi_0$ we then have

$$\begin{aligned} K_1(\phi)-K_1(\phi_0) &= \sum_{k=1}^{\infty}\gamma_k t^k\left\{|\nabla v|^p\left(\frac{A_1(x)\nabla v\cdot\nabla v}{|\nabla v|^2}\right)^k - |\nabla v_0|^p\left(\frac{A_1(x)\nabla v_0\cdot\nabla v_0}{|\nabla v_0|^2}\right)^k\right\} \\ &:= \sum_{k=1}^{\infty}\gamma_k t^k\Big\{|\nabla v|^p a(v)^k - |\nabla v_0|^p(a(v_0)^k\Big\}. \end{aligned}$$

We write

$$B(x) := |\nabla v|^p a(v)^k - |\nabla v_0|^p(a(v_0)^k = (|\nabla v|^p - |\nabla v_0|^p)a(v)^k + |\nabla v_0|^p(a(v)^k - a(v_0)^k).$$

As we showed in the first part we have $|\nabla v|^p, |\nabla v_0|^p \in C^{1,\alpha}_{\nu-2} \hookrightarrow C^{0,\alpha}_{\nu-2}$ (continuously). Also for $r$ sufficiently small we have $\frac{|\nabla v|^2-|\nabla v_0|^2}{|\nabla v_0|^2} < 1$ then we can write

$$|\nabla v|^p - |\nabla v_0|^p = |\nabla v_0|^p\Big((1+\frac{|\nabla v|^2-|\nabla v_0|^2}{|\nabla v_0|^2})^{\frac{p}{2}} - 1\Big) = \sum_{k=1}^{\infty}\gamma_k(\frac{|\nabla v|^2-|\nabla v_0|^2}{|\nabla v_0|^2})^k.$$

Note that we have

$$b(x) := \frac{|\nabla v|^2-|\nabla v_0|^2}{|\nabla v_0|^2} = \frac{\nabla(v+v_0)\cdot\nabla(\phi-\phi_0)}{|\nabla v_0|^2}.$$

Then for $r$ small

$$\|b\|_{C^{0,\alpha}_{\nu-2}} \le C\|\phi-\phi_0\|_{C^{2,\alpha}_{\nu}} \le 2Cr$$

and so we get

$$\||\nabla v|^p - |\nabla v_0|^p\|_{C^{0,\alpha}_{\nu-2}} \leq \sum_{k=1}^{\infty} |\gamma_k|(C\|\phi - \phi_0\|_{C^{2,\alpha}_\nu})^k$$
$$\leq C\|\phi - \phi_0\|_{C^{2,\alpha}_\nu} \sum_{k=1}^{\infty} |\gamma_k|(2Cr)^{k-1} \leq C_1\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}.$$

Also as it is shown in the first part we have $a(v), a(v_0) \in C^{1,\alpha}_0$, and using again the formula $a^k - b^k = (a-b)\sum_{i=0}^{k-1} a^i b^{k-1-i}$ we can write

$$a(v)^k - a(v_0)^k = (a(v) - a(v_0))\sum_{i=0}^{k-1} a(v)^i a(v_0)^{k-1-i}.$$

Now, similar as we did above, it is really not hard to see that

$$\|a(v) - a(v_0)\|_{C^{1,\alpha}_0} \leq c\|\nabla(v - v_0)\|_{C^{1,\alpha}_{\nu-1}} = c\|\nabla(\phi - \phi_0)\|_{C^{1,\alpha}_{\nu-1}} \leq c\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}.$$

Then taking $C \geq \max\{\|a(v)\|_{C^{1,\alpha}_0}, \|a(v_0)\|_{C^{1,\alpha}_0}\}$ we get

$$\|a(v)^k - a(v_0)^k\|_{C^{1,\alpha}_0} \leq ckC^{k-1}\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}.$$

Using all the above obtained estimates we get

$$\|B(x)\|_{C^{0,\alpha}_{\nu-2}} \leq kC^{k-1}\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}.$$

Hence,

$$\|K_1(\phi) - K_1(\phi_0)\|_{C^{0,\alpha}_{\nu-2}} \leq \sum_{k=1}^{\infty} |\gamma_k| t^k kC^{k-1}\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}$$
$$= t\Big(\sum_{k=1}^{\infty} |\gamma_k| k(tC)^{k-1}\Big)\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}.$$

Thus taking $t$ small such that $k(tC)^{k-1} < (\frac{1}{2})^k$ for $k \geq 2$, we get

$$\|K_1(\phi) - K_1(\phi_0)\|_{C^{0,\alpha}_{\nu-2}} \leq Ct\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}. \tag{28}$$

Finally using (27), (28), the fact that $\|E_t(\phi - \phi_0)\|_{C^{0,\alpha}_{\nu-2}} \leq Ct\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}$ and continuity of $L^{-1}$ we get, for small $t$ and $r$,

$$\|J_t(\phi) - J_t(\phi_0)\|_{C^{0,\alpha}_{\nu-2}} \leq (Ct + C(r))\|\phi - \phi_0\|_{C^{2,\alpha}_\nu}, \quad C(r) \to 0 \text{ as } r \to 0.$$

This shows that for sufficiently small $t$ and $r$, $J_t : B_r \to B_r$ is a contraction and hence we can apply Banach's Contraction Mapping Principle to obtain a fixed point $\phi \in B_r$. Now recall that

$v(x) = w(x) + \phi(x)$ and note that $0 < w \in C^{2,\alpha}_{\nu,D}$ and hence for $r > 0$ small enough we have $v(x) > 0$ for $0 < |x| < 1$ after taking into account the growth of the functions near $x = 0$ and the fact that $w'(1) < 0$. Moreover we have $v(x) \to \infty$ as $|x| \to 0$. Indeed, we have $v(x) \approx C|x|^{\frac{p-2}{p-1}}$ near the origin. To see this, note that we have $\|v - w\|_{C^{2,\alpha}_\nu} = \|\phi\|_{C^{2,\alpha}_\nu} < r$. This in particular gives $\sup_{0<s<\frac{1}{4}} s^{-\nu} \sup_{A_s} |v(x) - w(x)| < r$. Hence, $|v(x) - w(x)| < r|x|^\nu$ for $0 < |x| < \frac{1}{2}$. Note that we have $w(x) = C_p (|x|^\nu - 1)$ (and $\nu < 0$) thus taking $r$ sufficiently small we get $c_1|x|^\nu \leq v(x) \leq c_2|x|^\nu$ for $|x|$ small. Taking into account other terms in the definition of $\|v - w\|_{C^{2,\alpha}_\nu}$ we obtain similar estimates on $\nabla v$ and $D^2 v$, and see that $v$ behaves like $|x|^\nu$ near the origin. This proves the second part of the first assertion in Theorem 1, noting that $u(y) = v(x)$ where $x$ and $y$ are related through $y = x + t\psi(x)$.

## 3. Case $p > 2$

In this section we will always assume

$$N \geq 3, \quad p > 2, \quad \tau := \frac{p-2}{p-1}, \quad \nu := \tau. \tag{29}$$

We are now interested in obtaining positive nonclassical solutions of (4) in the case of $p > 2$ for $t > 0$ small enough. Recall from Example 1 that $w(x) := C_p (1 - |x|^\tau)$, where $C_p^{p-1} := \frac{N-2+\tau}{\tau^{p-1}}$, is a $C^{0,\tau}(\overline{B_1})$ weak solution of (5) in the case of $t = 0$. So we will look for solutions of (5), in the case of $0 < t$ small, in the form of $v(x) = w(x) + \phi(x)$ where $\phi$ is in a suitable space. As in the case of $\frac{N}{N-1} < p < 2$ we need $\phi$ to satisfy (6) and to find a $\phi$ we will apply the Banach Contraction Mapping Principle to the nonlinear mapping $J_t$ defined as in (7), which at this point is not well defined. As before a crucial step will be to understand the linearized operator $L(\phi) := -\Delta\phi + -p|\nabla w|^{p-2}\nabla w \cdot \nabla\phi$ associated with the explicit radial solution $w$. A computation shows $L$ is given by

$$L(\phi) := -\Delta\phi + \frac{p(N-2+\tau)(x \cdot \nabla\phi(x))}{|x|^2}.$$

The space we will work on is $C^{2,\alpha}_{\nu,D}(\overline{B}\backslash\{0\})$.

**Theorem** *([34], Proposition 2.3). Suppose $N, p, \tau, \nu$ are as in (29). There is some $C$ such that for all $f \in C^{0,\alpha}_{\nu-2}$ there is unique $\phi \in C^{2,\alpha}_\nu$ and $\zeta \in \mathbb{R}$ such that $-\Delta\phi = f$ in $B_1\backslash\{0\}$ with $\phi = \zeta$ on $\partial B_1$ (note we are not prescribing $\zeta$, it depends on $f$). Additionally we have $\|\phi\|_{C^{2,\alpha}_\nu} \leq C\|f\|_{C^{0,\alpha}_{\nu-2}}$.*

**Corollary 1.** *Suppose $N, p, \tau, \nu$ are as in (29) and suppose $f \in C^{0,\alpha}_{\nu-2}$ has no $k = 0$ mode. Let $\phi, \zeta$ be from the above theorem. Then $\zeta = 0$ and $\phi$ has no $k = 0$ mode.*

So in this new parameter range for $\nu$ it is only the $k = 0$ mode that is causing any issues.

**Proof.** We write $\phi(x) = \sum_{k=0}^\infty a_k(r)\Theta_k(\theta)$ and note that since $f$ has no $k = 0$ mode we have $a_0(r) = \frac{C_1}{r^{N-2}} + C_2$ and after taking into account the behavior of $\phi$ near $x = 0$ we see that $C_1 = 0$. So $\phi(r\theta) = \phi(x) = C_2 + \sum_{k=1}^\infty a_k(r)\Theta_k(\theta)$ and integrating this over $S^{N-1}$ we arrive at

$C_2|S^{N-1}| = \int_{S^{N-1}} \phi(r\theta)d\theta$ and hence this quantity is independent of $0 < r < 1$. But recall that $|\phi(x)| \le C|x|^\tau$ and hence by sending $r \searrow 0$ we have $C_2 = 0$ and now sending $r \nearrow 1$ we have $\zeta = 0$. □

**Definition 2.** Let $N, p, \tau, \nu$ be as in (29) and set

$$Y^\alpha_{\nu-2} := Y = \left\{ f \in C^{0,\alpha}_{\nu-2}(B_1\backslash\{0\}) : \ f \text{ has no } k = 0 \text{ mode} \right\},$$

and

$$X^\alpha_{\nu,D} := X = \left\{ \phi \in C^{2,\alpha}_{\nu,D}(B_1\backslash\{0\}) : \ \phi \text{ has no } k = 0 \text{ mode} \right\},$$

and we use the natural norm the subspace inherits from the full space.

So $\Delta : X \to Y$ is an isomorphism.

**Proposition 2.** *Let $N, p, \tau, \nu$ be as in (29). Then there is some $C > 0$ such that for all $f \in Y^\alpha_{\nu-2}$ there is some $\phi \in X^\alpha_{\nu,D}$ such that $L(\phi) = f$ in $B_1\backslash\{0\}$ with $\phi = 0$ on $\partial B_1$. Moreover one has the estimate $\|\phi\|_{C^{2,\alpha}_\nu} \le C\|f\|_{C^{0,\alpha}_{\nu-2}}$.*

**Proof.** We begin by analyzing the kernel of $L_\gamma$ on the unit ball and also the full space.

Suppose $\phi \in X$ satisfies $L_\gamma(\phi) = 0$ in $B_1\backslash\{0\}$. As in Lemma 2 we write $\phi = \sum_{k=1}^\infty a_k(r)\Theta_k(\theta)$ and so $a_k$ satisfies

$$a_k''(r) + \frac{(N - 1 - \gamma p(N - 2 + \tau))}{r} a_k'(r) - \frac{\lambda_k}{r^2} a_k(r) = 0 \quad 0 < r < 1, \tag{30}$$

with $a_k(1) = 0$ for $k \ge 1$. As before this is an equation of Euler form and so we look for solutions of the form $r^\alpha$ and so $\alpha$ must satisfy $\alpha^2 + \alpha b - \lambda_k = 0$ where $b = b_\gamma = N - 2 - \gamma p(N - 2 + \tau)$ and as before this has roots

$$\alpha_k^- := \frac{-b}{2} - \frac{\sqrt{b^2 + 4\lambda_k}}{2}, \qquad \alpha_k^+ := \frac{-b}{2} + \frac{\sqrt{b^2 + 4\lambda_k}}{2}.$$

Taking into the boundary condition we see we must have $a_k(r) = C_k(r^{\alpha_k^+} - r^{\alpha_k^-})$. Note first that $\alpha_k^+ > 0$ and also note that since $\alpha_k^+ \ne \alpha_k^-$ we see that if $\alpha_k^- < \tau$ then we must have $C_k = 0$ otherwise $\phi \notin X$. But note $\alpha_k^- < 0$ and so we have $a_k = 0$ for $k \ge 1$ and hence $\phi = 0$.

We now consider the full space problem; which we will arrive at later after a blow up argument. Let $\psi \in C^\infty(\mathbb{R}^N\backslash\{0\})$ which satisfies

$$-\Delta\psi + \frac{\gamma p(N - 2 + \tau)(x \cdot \nabla\psi)}{|x|^2} = 0 \quad \mathbb{R}^N\backslash\{0\},$$

and we also assume $\psi$ satisfies the pointwise bound $|\psi(x)| \le C|x|^\tau$ for some $C > 0$. Additionally we assume $\psi$ has no $k = 0$ mode. We now show $\psi = 0$; as before we assume $\psi$ we can write $\psi(x) = \sum_{k=1}^\infty a_k(r)\Theta_k(\theta)$. Then $a_k$ satisfies (30) on $0 < r < \infty$ and so we have $a_k(r) = C_k r^{\alpha_k^+} - D_k r^{\alpha_k^-}$. Again since $\alpha_k^+ > \alpha_k^-$ and $\alpha_k - < 0$ (and hence less than $\tau$)

we have $D_k = 0$. By using the pointwise bound on $\psi$ at both the origin and infinity we see that we must have $C_k = 0$ unless $\alpha_k^+ = \tau$. We now rule out this case; so we need to rule out $h := -b + \sqrt{b^2 + 4\lambda_k} = 2\tau$. First note that $h'(b) < 0$ and so we have

$$h(b(\gamma)) \geq h(\max_{\gamma \in [0,1]} b(\gamma)) = h(b(0)) = h(N-2) \geq 2 > \frac{2(p-2)}{p-1},$$

and hence we have $-b + \sqrt{b^2 + 4\lambda_k} > 2\tau$ for all $\gamma \in [0, 1]$ and $k \geq 1$. We can now conclude that $\psi = 0$.

We now examine the operator $L$ and as before we use a continuation argument; for $0 \leq \gamma \leq 1$ set $L_\gamma(\phi) := -\Delta\phi + \frac{p(N-2+\tau)(x \cdot \nabla\phi(x))}{|x|^2}$. To begin we need to show that $(\gamma, \phi) \mapsto L_\gamma(\phi)$ is a continuous linear operator from $[0, 1] \times X_\nu^\alpha$ to $Y_{\nu-2}^\alpha$. The proof of this result for the previous range of $p$ carries over to this range of $p$. So to prove the desired estimate for $L$ it is sufficient to prove a priori estimates independent of $\gamma$. So towards a contradiction we assume $L_{\gamma_m}(\phi_m) = f_m$ where $\|f_m\|_Y \to 0$ and $\|\phi_m\|_X = 1$ (here $Y = Y_{\nu-2}^\alpha$ and $X = X_\nu^\alpha$) and $\gamma_m \in [0, 1]$. We now derive a contradiction. The proof is almost exactly the same as in the case of the other range of $p$. The only issue one needs to be careful with is that we don't have the needed theory for $\Delta$ on $C_{\nu,D}^{2,\alpha}$ to $C_{\nu-2}^{0,\alpha}$ now; but we have the theory on $X_\nu^\alpha$ to $Y_{\nu-2}^\alpha$ and this suffices. We now argue exactly as before by considering the various cases. In each case we obtain the needed contradiction after using the above results regarding the kernel of $L_\gamma$ on the unit ball and the full space. □

We now obtain the needed linear theory for the $k = 0$ mode.

**Lemma 7** *(Mode $k = 0$). Let $N, p, \tau, \nu$ be as in (29). Then there is some $C > 0$ such that for all $f = f(r)$ there is some $a = a(r)$ such that $L(a) = -f$ in $B_1 \backslash \{0\}$ with $a(1) = 0$. Moreover we have $\|a\|_{C_\nu^{2,\alpha}} \leq C\|f\|_{C_{\nu-2}^{0,\alpha}}$.*

**Proof.** Note $a$ must satisfy $a''(r) + \frac{\beta a'(r)}{r} = f$ where $\beta := N - 1 - p(N - 2 + \tau)$. Using the integrating factor method and integrating between $r$ and 1 we arrive at (after setting $a'(1) = T$ a free parameter to be picked later)

$$r^\beta a'(r) = T - \int_r^1 t^\beta f(t)dt.$$

Note we have $t^{2-\tau}|f(t)| \leq \|f\|_{C_{\nu-2}^{0,\alpha}}$. From this note we have

$$r^\beta |a'(r)| \leq |T| + \int_r^1 \frac{\|f\|_{C_{\nu-2}^{0,\alpha}}}{t^{2-\tau-\beta}},$$

and a computation shows we have $2 - \tau - \beta > 1$ and so we have

$$r^\beta |a'(r)| \leq |T| + C\|f\|_{C_{\nu-2}^{0,\alpha}} r^{\beta-1+\tau},$$

and hence we have

$$r^{1-\tau}|a'(r)| \le |T| r^{1-\tau-\beta} + C\|f\|_{C^{0,\alpha}_{\nu-2}} \le |T| + C\|f\|_{C^{0,\alpha}_{\nu-2}},$$

since $1-\tau-\beta > 0$.

We now get a formula for $a$ and some estimates on $a$.

Writing out $a'(r)$ from our earlier formula and integrating $r \in (R, 1)$ we arrive at

$$-a(R) = T \int_R^1 r^{-\beta} dr - \int_R^1 r^{-\beta} \int_r^1 t^{\beta} f(t) dt dr,$$

after using the fact that $a(1) = 0$. We can use Fubini's theorem on the double integral to see

$$(1-\beta) \int_R^1 r^{-\beta} \int_r^1 t^{\beta} f(t) dt dr = \int_R^1 t f(t) dt - R^{1-\beta} \int_R^1 t^{\beta} f(t) dt.$$

So we can now write out

$$-(1-\beta) a(R) = T - C_f + \int_0^R t f(t) dt - T R^{1-\beta} + R^{1-\beta} \int_R^1 t^{\beta} f(t) dt,$$

where $C_f := \int_0^1 t f(t) dt$ (which is finite since $t f(t)$ is integrable on $(0, 1)$). We want $\frac{a(R)}{R^{\nu}}$ to be bounded for $0 < R$ small. So we will take $T = C_f$ and hence

$$-(1-\beta) a(R) = \int_0^R t f(t) dt - T R^{1-\beta} + R^{1-\beta} \int_R^1 t^{\beta} f(t) dt.$$

Note that

$$|T| \le \int_0^1 t|f(t)| dt \le \|f\|_{C^{0,\alpha}_{\nu-2}} \int_0^1 t^{\tau-1} dt \le C_{\tau} \|f\|_{C^{0,\alpha}_{\nu-2}}.$$

We can compute the various estimates for $a$ directly from the formula for $a$. □

**Lemma 8** *(Combining the linear estimates). Let $N, p, \tau, \nu$ be as in (29). Then there is some $C > 0$ such that for all $f \in C^{0,\alpha}_{\nu-2}(\overline{B_1}\backslash\{0\})$ there is some $\phi \in C^{2,\alpha}_{\nu,D}(\overline{B_1}\backslash\{0\})$ such that $L(\phi) = f$ in $B_1\backslash\{0\}$ with $\phi = 0$ on $\partial B_1$. Moreover one has $\|\phi\|_{C^{2,\alpha}_{\nu}} \le C\|f\|_{C^{0,\alpha}_{\nu-2}}$.*

**Proof.** The proof is the same as the case of $p < 2$. □

**The fixed point argument.** We proceed exactly as we did in the case of $\frac{N}{N-1} < p < 2$. For sufficiently small $t > 0$ and $r > 0$ we are able to find a $\phi \in B_r \subset C^{2,\alpha}_{\nu,D}(\overline{B_1}\backslash\{0\})$ (and recall $\nu = \tau$) which satisfies (6) and hence $v(x) = w(x) + \phi(x)$ satisfies (5). We now recall that $w(x) := C_p\,(1 - |x|^\tau)$ where $\tau = \frac{p-2}{p-1}$ and where $C_p > 0$. Arguing as in the case of $\frac{N}{N-1} < p < 2$ we see that by taking $t, r > 0$ sufficiently small we have $v > 0$ in $B_1$. Also similar as in the previous section, from the inequality $\|v - w\|_{C^{2,\alpha}_\nu} = \|\phi\|_{C^{2,\alpha}_\nu} < r$, we get $|v(x) - w(x)| < r|x|^\tau$ for $0 < |x| < \frac{1}{2}$, and since $\tau > 0$ this gives $v(0) = C_p$. Then we can write the later inequality as $|v(x) - v(0) + C_p|x|^\tau| < r|x|^\tau$, implies that $\frac{|v(x)-v(0)|}{|x|^\tau} \geq C_p - r > 0$ for $r$ small. This in particular gives

$$\lim_{x\to 0} \frac{|v(x) - v(0)|}{|x|^{\tau+\varepsilon}} = \infty,$$

which shows that $v \notin C^{0,\tau+\varepsilon}(\overline{B_1})$ for any $\varepsilon > 0$. Recalling that $u(y) = v(x)$ where $x$ and $y$ are related through $y = x + t\psi(x)$. Since $t > 0$ is small and $\psi$ is smooth this gives that $u \notin C^{0,\tau+\varepsilon}(\overline{\Omega_t})$ for any $\varepsilon > 0$.

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## References

[1] D. Arcoya, L. Boccardo, T. Leonori, A. Porretta, Some elliptic problems with singular natural growth lower order terms, J. Differential Equations 249 (2010) 2771–2795.
[2] D. Arcoya, J. Carmona, T. Leonori, P.J. Martinez-Aparicio, L. Orsina, F. Petitta, Existence and non-existence of solutions for singular quadratic quasilinear equations, J. Differential Equations 246 (2009) 4006–4042.
[3] D. Arcoya, C. De Coster, L. Jeanjean, K. Tanaka, Remarks on the uniqueness for quasilinear elliptic equations with quadratic growth conditions, J. Math. Anal. Appl. 420 (2014) 772–780.
[4] D. Arcoya, C. De Coster, L. Jeanjean, K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient, J. Funct. Anal. 268 (2015) 2298–2335.
[5] A. Aghajani, C. Cowan, S.H. Lui, Existence and regularity of nonlinear advection problems, Non. Anal. 166 (2018) 19–47.
[6] A. Bensoussan, L. Boccardo, F. Murat, On a nonlinear partial differential equation having natural growth terms and unbounded solution, Ann. Inst. Henri Poincaré 5 (1988) 347–364.
[7] M.-F. Bidaut-Veron, M. Garcia-Huidobro, L. Veron, Remarks on some quasilinear equations with gradient terms and measure data, Contemp. Math. 595 (2013) 31–53.
[8] M.-F. Bidaut-Veron, M. Garcia-Huidobro, L. Veron, Local and global properties of solutions of quasilinear Hamilton–Jacobi equations, J. Funct. Anal. 267 (2014) 3294–3331.
[9] M.-F. Bidaut-Veron, M. Garcia-Huidobro, L. Veron, Boundary singularities of positive solutions of quasilinear Hamilton–Jacobi equations, Calc. Var. 54 (2015) 3471–3515.
[10] J. Ching, F.C. Cirstea, Existence and classification of singular solutions to nonlinear elliptic equations with a gradient term, Anal. PDE 8 (2015) 1931–1962.
[11] J.M. Coron, Topologie et cas limite des injections de Sobolev, C. R. Acad. Sci. Paris 299 (1984) 209–212, Series I.
[12] J. Dávila, M. del Pino, M. Musso, The supercritical Lane–Emden–Fowler equation in exterior domains, Comm. Partial Differential Equations 32 (8) (2007) 1225–1243.
[13] J. Dávila, M. del Pino, M. Musso, J. Wei, Fast and slow decay solutions for supercritical elliptic problems in exterior domains, Calc. Var. Partial Differential Equations 32 (4) (August 2008) 453–480.
[14] J. Dávila, Manuel del Pino, M. Musso, J. Wei, Standing waves for supercritical nonlinear Schrödinger equations, J. Differential Equations 236 (1) (2007) 164–198.

[15] J. Dávila, L. Dupaigne, Perturbing singular solutions of the Gelfand problem, Commun. Contemp. Math. 9 (5) (2007) 639–680.
[16] M. del Pino, M. Musso, Super-critical bubbling in elliptic boundary value problems, in: Variational Problems and Related Topics, Kyoto, 2002, 2003, pp. 85–108, 1307.
[17] M. del Pino, P. Felmer, Monica Musso, Two-bubble solutions in the super-critical Bahri–Coron's problem, Calc. Var. Partial Differential Equations 16 (2) (2003) 113–145.
[18] M. del Pino, P. Felmer, M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries, Bull. Lond. Math. Soc. 35 (4) (2003) 513–521.
[19] V. Ferone, F. Murat, Nonlinear problems having natural growth in the gradient: an existence result when the source terms are small, Nonlinear Anal. 42 (2000) 1309–1326.
[20] V. Ferone, M.R. Posteraro, J.M. Rakotoson, $L^\infty$-estimates for nonlinear elliptic problems with $p$-growth in the gradient, J. Inequal. Appl. 3 (1999) 109–125.
[21] M. Gherga, V. Radulescu, Nonlinear PDEs, Springer-Verlag, Berlin, Heidelberg, 2012.
[22] D. Giachetti, F. Petitta, S. Segura de Leon, Elliptic equations having a singular quadratic gradient term and a changing sign datum 11 (2012) 1875–1895.
[23] D. Giachetti, F. Petitta, S. Segura de Leon, A priori estimates for elliptic problems with a strongly singular gradient term and a general datum, Differential Integral Equations 226 (2013) 913–948.
[24] B. Gidas, J. Spruck, Global and local behavior of positive solutions of nonlinear elliptic equations, Comm. Pure Appl. Math. 34 (4) (1981) 525–598.
[25] D. Gilbarg, N. Trudinger, Elliptic Partial Differential Equations of Second Order, Classics in Mathematics, Springer-Verlag, Berlin, 2001.
[26] N. Grenon, F. Murat, A. Porretta, Existence and a priori estimate for elliptic problems with subquadratic gradient dependent terms, C. R. Acad. Sci. Paris, Ser. I 342 (2006) 23–28.
[27] N. Grenon, C. Trombetti, Existence results for a class of nonlinear elliptic problems with $p$-growth in the gradient, Nonlinear Anal. 52 (2003) 931–942.
[28] J.M. Lasry, P.L. Lions, Nonlinear elliptic equations with singular boundary conditions and stochastic control with state constraints, Math. Ann. 283 (1989) 583–630.
[29] P.L. Lions, Quelques remarques sur les problemes elliptiques quasilineaires du second ordre, J. Anal. Math. 45 (1985) 234–254.
[30] M. Marcus, P.T. Nguyen, Elliptic equations with nonlinear absorption depending on the solution and its gradient, Proc. Lond. Math. Soc. 111 (2015) 205–239.
[31] R. Mazzeo, F. Pacard, A construction of singular solutions for a semilinear elliptic equation using asymptotic analysis, J. Differential Geom. 44 (1996) 331–370.
[32] P.T. Nguyen, Isolated singularities of positive solutions of elliptic equations with weighted gradient term, Anal. PDE 9 (7) (2016) 1671–1692.
[33] P.T. Nguyen, L. Veron, Boundary singularities of solutions to elliptic viscous Hamilton–Jacobi equations, J. Funct. Anal. 263 (2012) 1487–1538.
[34] F. Pacard, T. Rivière, Linear and Nonlinear Aspects of Vortices: The Ginzburg Landau Model, Progress in Nonlinear Differential Equations, vol. 39, Birkäuser, 2000, 342 pp.
[35] D. Passaseo, Nonexistence results for elliptic problems with supercritical nonlinearity in nontrivial domains, J. Funct. Anal. 114 (1) (1993) 97–105.
[36] S. Pohozaev, Eigenfunctions of the equation $\Delta u + \lambda f(u) = 0$, Sov. Math., Dokl. 6 (1965) 1408–1411.
[37] A. Porretta, S. Segura de Leon, Nonlinear elliptic equations having a gradient term with natural growth, J. Math. Pures Appl. 85 (2006) 465–492.
[38] M. Struwe, Variational Methods – Applications to Nonlinear Partial Differential Equations and Hamiltonian Systems, Springer-Verlag, Berlin, 1990.
[39] Z. Zhang, Boundary blow-up elliptic problems with nonlinear gradient terms, J. Differential Equations 228 (2006) 661–684.