On supercritical elliptic problems: existence, multiplicity of positive and symmetry breaking solutions*

Craig Cowan† Abbas Moameni‡

Abstract

The main thrust of our current work is to exploit very specific characteristics of a given problem in order to acquire improved compactness for supercritical problems and to prove existence of new types of solutions. To this end, we shall develop a variational machinery in order to construct a new type of classical solutions for a large class of supercritical elliptic partial differential equations.
The issue of symmetry and symmetry breaking is challenging and fundamental in mathematics and physics. Symmetry breaking is the source of many interesting phenomena namely phase transitions, instabilities, segregation, etc. As a consequence of our results we shall establish the existence of several symmetry breaking solutions when the underlying problem is fully symmetric. Our methodology is variational, and we are not seeking non symmetric solutions which bifurcate from the symmetric one. Instead, we construct many new positive solutions by developing a minimax principle for general semilinear elliptic problems restricted to a given convex subset instead of the whole space. As a byproduct of our investigation, several new Sobolev embeddings are established for functions having a mild monotonicity on symmetric monotonic domains.

2010 Mathematics Subject Classification: 35J15, 35A15, 35A16, 35B07

Key words: Supercritical elliptic equations, Variational and topological methods.

Contents

*Both authors are pleased to acknowledge the support of the National Sciences and Engineering Research Council of Canada.

†University of Manitoba, Winnipeg Manitoba, Canada, craig.cowan@umanitoba.ca

‡School of Mathematics and Statistics, Carleton University, Ottawa, Ontario, Canada, momeni@math.carleton.ca

1 Introduction

In this work we develop a variational machinery to examine a large class of significant supercritical elliptic partial differential equations that arise naturally in various physical models: solitary waves in nonlinear Schrödinger equations; gravitational potential of a Newtonian self gravitating, spherically symmetric, polytropic fluid; and a model for a cluster of stars. Our method is variational but as opposed to working on the natural energy space, which typically limits problems to subcritical and critical, we work on closed convex sets (not necessarily a linear subspace) which increases the available compactness. Working on symmetric functions can sometimes increase compactness, which together with the principal of symmetric criticality provides an efficient tool to deal seemingly nocompact settings (see for instance [5] and [46]). Our method further increases compactness as we are restricting our problems on an appropriate subsets which goes well beyond the symmetry induced function spaces under certain compact groups. The main thrust of our current work is to exploit very specific characteristics of a given problem in order to acquire improved compactness for supercritical problems and to prove existence of new types of solutions. Our approach is broad enough to cover many elliptic partial differential equations, and in general, one can employ a combination of symmetry, monotonicity, smallness in certain norms, convexity, and etc to name a few.

Broadly speaking we are interested in obtaining positive classical solutions of equations of the form

$$-\Delta u + V(x)u = a(x)u^{p-1} \text{ in } \Omega, \tag{1}$$

where $\Omega \subset \mathbb{R}^N$ is either the full space or Ω is a bounded subset and in which case we add the boundary condition $u = 0$ on $\partial\Omega$. Our main interest will be in obtaining solutions in the case of $p > 2$ and supercritical. Generally a will be a sufficiently smooth function which satisfies some symmetry and monotonicity assumptions and we point out any added compactness is not coming from a; which is a different phenomena from the Hénon equation. The domains we will examine will be domains of double and triple revolution with some added monotonicity properties. Additionally when the problems has extra symmetry we will obtain solutions which do not inherit the extra symmetry of the problem. On radial domains we will obtain nonradial solutions which are not foliated Schwarz symmetric. As a consequence of our approach, many new multiplicity results are also obtained.

Since we address existence and multiplicity issues for numerous supercritical problems we list the equations here for the convenience of the readers. Even though each of these problems poses their own difficulty, our variational machinery is able to give a unified approach.

- In Section 4 we examine the following problem

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Here we consider annular domains which are radial and nonradial. On the radial domains we obtain new type of positive nonradial solutions for which do not have the foliated Schwarz symmetry. In all cases we obtain results for a supercritical range of p. The main result is Theorem 4.1.

- In Section 5 we examine

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{3}$$

where B_1 is the unit ball in $\mathbb{R}^N$. In Theorem 5.1, we obtain several types of positive new nonradial solutions on a range of supercritical p.

- In Section 6 we examine

$$-\Delta u + u = |x|^\alpha u^{p-1} \qquad \text{in } \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^n, \tag{4}$$

and we show there is a positive classical solution for

$$\frac{2N+2\alpha-4}{N-2} < p < \frac{2N+2\alpha}{N-2},$$

and for large α we obtain a nonradial solution. Theorem 6.1 is devoted to this problem.

- In Section 7 we examine

$$\begin{cases} -\Delta u + \frac{u}{|x|^\alpha} = u^{p-1} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{5}$$

where $\alpha > 2$ (note this is in some sense supercritical). For $2 < p < \frac{2N+2\alpha-4}{N-2}$ we obtain a positive classical solution of (5) and for large α we obtain a nonradial solution. Additionally the solution decays to zero at the origin quicker than any polynomial. See Theorem 7.1 for details. Note here the zero order potential is playing a key role and we believe this is new phenomena.

- In Section 8 we give an approach to show ground states of various problems on radial domains are nonradial. Indeed, as stated in Theorem 8.1, the best constant in the well known hardy inequality corresponding to the underlying domain plays a major role to address this challenging affair.

- In Section 9 we examine

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where Ω is a bounded domain in $\mathbb{R}^N$ which is also a domain of triple revolution. Under various assumptions we prove the existence and also some multiplicity results (for a range of supercritical p). We have listed our contributions in Theorems 9.1, 9.4, 9.5 and Corollary 9.5.

A crucial step in proving the above existence results will be in obtaining improved Sobolev imbeddings for various classes of symmetric and monotonic functions. The increases in compactness comes from two distinct properties of the closed convex sets we choose to work on, namely the symmetry and also the monotonicity. One should note that these improved imbeddings also play a crucial role in the proof of the regularity of the solution. One added benefit of our approach is we can use energy levels directly to prove various results.

1.1 Outline of the paper

We now give a brief outline of the paper. In Section 2 we develop our abstract variational machinery. In Section 3 we introduce domains of m revolution and in particular we discuss domains of double revolution. Then in Section 4 we consider elliptic problems on domains of double revolution which are also annular type domains. The Hénon equation on the unit ball is considered in Section 5. In Section 6 we consider a Hénon like equation, but with a zero order term, on the full space. In Section 7 we consider a singular potential problem. Section 8 is where we develop the needed machinery to obtain solutions on symmetric domains without the naturally expected symmetry. Finally in Section 9 we consider domains of triple revolution.

1.2 Background

Here we give some background on the the problem and for this we take $a(x) = 1$ and $V(x) = 0$ and hence we consider

$$\begin{cases} -\Delta u = u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

We assume Ω a bounded smooth domain in $\mathbb{R}^N$. For $N \geq 3$ the critical exponent $2^* := \frac{2N}{N-2}$ plays a crucial role and for $2 < p < 2^*$ a variational approach shows the existence of a smooth positive solution of (7). For $p \geq 2^*$ there is no positive classical solution via the Pohozaev identity on star shaped domains, see [57]. For general domains in the critical/supercritical case, $p \geq 2^*$, the existence versus nonexistence of positive solutions of (7) presents a great degree of difficulties; see [6, 22, 32, 31, 30, 29, 33, 49, 53, 54, 59, 60]. Many of these results are very technical and some require perturbation arguments.

The possibility of utilizing the most of features that a given problem can offer to gain improved compactness for supercritical problems and to prove existence of new types of solutions is what motivated us for this work. As mentioned earlier, these features could be a combination of symmetry, monotonicity, convexity and etc. For instance, let us consider the Neumann boundary problem

$$\begin{cases} -\Delta u + u = a(r)u^{p-1} & \text{in } B_1, \\ \partial_\nu u = 0 & \text{on } \partial B_1, \end{cases} \tag{8}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$. The interest here is in obtaining nontrivial solutions for values of $p > \frac{2N}{N-2}$. In [8] they considered the variant of (8) given by $-\Delta u + u = |x|^\alpha u^{p-1}$ in B_1 with $\frac{\partial u}{\partial \nu} = 0$ on ∂B_1 (for Dirichlet versions of the Hénon equation see, for instance, [52, 36, 24]). They proved the existence of a positive radial solutions of this equation with arbitrary growth using a shooting argument. The solution turns out to be an increasing function. They also perform numerical computations to see the existence of positive oscillating solutions. In [61] they considered (8) along with the classical energy associated with the equation given by

$$E(u) := \int_{B_1} \frac{|\nabla u|^2 + u^2}{2}\, dx - \int_{B_1} a(|x|)F(u)\, dx,$$

where $F'(u) = f(u)$ (they considered a more general nonlinearity). Their goal was to find critical points of E over $H^1_{rad}(B_1) := \{u \in H^1(B_1) : u \text{ is radial}\}$. Of course since f is supercritical the standard approach of finding critical points will present difficulties and hence their idea was to find critical points of E over the cone $\{u \in H^1_{rad}(B_1) : 0 \leq u,\ u \text{ increasing}\}$. Doing this is somewhat standard but now the issue is the critical points don't necessarily correspond to critical points over $H^1_{rad}(B_1)$ and hence one can't conclude the critical points solve the equation; for instance the critical point could lie on the boundary of the convex cone and then one cannot perturb in all directions. The majority of their work was to show that in fact the critical points of E on the cone are really critical points over the full space. We remark that this work generated a lot of interest in this equation and many authors investigated these idea's of using monotonicity to overcome a lack of compactness. For further results regarding these Neumann problems on radial domains (some using these monotonicity ideas and some using other new methods) see [3, 39, 10, 9, 11, 21, 27, 47].

In [25], by making use of duality theory in convex analysis, we examined the super critical Neumann problem given by

$$\begin{cases} -\Delta u + u = a(x)f(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{9}$$

for multiradial domains which are a natural extension of radial domains. The idea of using convexity to deal with partial differential equations has a very long history starting from [35, 64] and also the recent papers [50, 51]. For Neumann problems on general domains see [3, 28, 38, 40, 41, 42, 43, 58, 65].

We now return to the Dirichlet problems. There have been many supercritical works that deal with domains that have certain symmetry, for instance, see [15, 16, 17, 18, 19, 20, 48].

In the case of the annulur domains the authors in [13, 14, 45] examined subcritical or slightly supercritical problems on expanding annuli and obtained nonradial solutions. In [37] they obtain nonradial solutions to supercritical problems on expanding annulur domains. In [7] they consider nonradial expanding annulur domains and they obtain the existence of positive solutions. In [33, 19] they consider domains with a small hole and obtain positive solutions. We shall also refer the interested reader to the recent works [1, 12, 26] where the idea of monotonicity together with variational and non-varitioanal methods were employed to deal with equation (9) in annular type domains.

2 A variational approach towards supercritical problems

In this section we assume that Ω is a domain in $\mathbb{R}^N$ which is not necessarily bounded. We also assume that a is a non-negative measurable function that is not identically zero. For $p > 1$, we define

$$L^p_a(\Omega) = \left\{ u : \int_\Omega a(x)|u|^p \, dx < \infty \right\},$$

equipped with the norm

$$\|u\|_{L^p_a(\Omega)} = \left(\int_\Omega a(x)|u|^p \, dx \right)^{\frac{1}{p}}.$$

We have the following general variational principle for possibly super critical elliptic problems.

Theorem 2.1. *(K ground state solution) Let Ω be a domain in $\mathbb{R}^N$, $p > 2$, and a be a non-negative function that is not identically zero. Let λ be a non-negative number which is strictly positive if Ω is unbounded. Consider the problem*

$$\begin{cases} -\Delta u + \lambda u = a(x)|u|^{p-2}u, & x \in \Omega, \\ u \in H^1_0(\Omega), \end{cases} \tag{10}$$

and its formal Euler-Lagrange functional

$$I(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx - \frac{1}{p}\int_\Omega a(x)|u|^p\,dx.$$

Let K be a convex and closed subset of $H_0^1(\Omega)$. Suppose the following two assertions hold:

(*i*) *K is compactly embedded in $L_a^p(\Omega)$, i.e., every bounded sequence in K has a converging subsequence in $L_a^p(\Omega)$.*

(*ii*) *(Pointwise invariance property) For each $\bar u \in K$ there exists $\bar v \in K$ such that*

$$-\Delta \bar v + \lambda \bar v = a(x)|\bar u|^{p-2}\bar u,$$

in the weak sense, i.e.,

$$\int_\Omega \nabla \bar v \cdot \nabla \eta\,dx + \lambda\int_\Omega \bar v \eta\,dx = \int_\Omega a(x)|\bar u|^{p-2}\bar u \eta\,dx, \qquad \forall \eta \in H_0^1(\Omega)\cap L_a^p(\Omega).$$

Then there exist $c>0$ and $\tilde u \in K$ such that $I(\tilde u) = c$ and $\tilde u$ is a weak solution of the equation

$$\begin{cases} -\Delta u + \lambda u = a(x)|u|^{p-2}u, & x\in\Omega \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{11}$$

*We call $\tilde u$ a K-**ground state solution** of (11). A characterization for the critical value c is given in the proof.*

We shall need some preliminaries before proving this theorem. Consider the Banach space $V = H_0^1(\Omega)\cap L_a^p(\Omega)$ equipped with the following norm

$$\|u\|_V = \|u\|_{H_0^1(\Omega)} + \|u\|_{L_a^p(\Omega)},$$

and note that the duality pairing between V and its dual V^* is defined by

$$\langle u, u^*\rangle = \int_\Omega u(x)u^*(x)\,dx, \qquad \forall u \in V,\, \forall u^* \in V^*.$$

We define $\Psi : V \to \mathbb{R}$ and $\Phi : V \to \mathbb{R}$ by

$$\Psi(u) = \frac{1}{2}\int_\Omega (|\nabla u|^2 + \lambda u^2)\,dx,$$

and

$$\Phi(u) = \frac{1}{p}\int_\Omega a(x)|u|^p dx.$$

We remark that even though Φ is not even well-defined on $H_0^1(\Omega)$ for large p, but it is continuously differentiable on the space $V = H_0^1(\Omega)\cap L_a^p(\Omega)$. Finally, let us introduce the functional $E_K(u) : V \to (-\infty, +\infty]$ defined by

$$E_K(u) := \Psi_K(u) - \Phi(u) \tag{12}$$

where

$$\Psi_K(u) = \begin{cases} \Psi(u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \tag{13}$$

Note that E_K is indeed the Euler-Lagrange functional corresponding to (11) restricted to K. We shall now recall some notations and results for the minimax principles for lower semi-continuous functions.

Definition 2.1. *Let V be a real Banach space, $\Phi \in C^1(V,\mathbb{R})$ and $\Psi : V \to (-\infty,+\infty]$ be proper (i.e. $Dom(\Psi) \neq \emptyset$), convex and lower semi-continuous. A point $u \in V$ is said to be a critical point of*

$$I := \Psi - \Phi \tag{14}$$

if $u \in Dom(\Psi)$ and if it satisfies the inequality

$$\langle D\Phi(u), u - v\rangle + \Psi(v) - \Psi(u) \geq 0, \qquad \forall v \in V, \tag{15}$$

where $\langle \cdot,\cdot\rangle$ is the duality pairing between V and its dual V^*.

Definition 2.2. *We say that I satisfies the Palais-Smale compactness condition (PS) if every sequence $\{u_n\}$ such that*

- $I[u_n] \to c \in \mathbb{R}$,
- $\langle D\Phi(u_n), u_n - v\rangle + \Psi(v) - \Psi(u_n) \geq -\varepsilon_n \|v - u_n\|, \qquad \forall v \in V,$

where $\varepsilon_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

The following non-smooth mountain pass theorem is due to A. Szulkin [63].

Theorem 2.2. *Suppose that $I : V \to (-\infty,+\infty]$ is of the form (14) and satisfies the Palais-Smale condition and the Moutaint Pass Geometry (MPG):*

1. *$I(0) = 0$.*
2. *There exists $e \in V$ such that $I(e) \leq 0$.*
3. *There exists some ρ such that $0 < \rho < \|e\|$ and for every $u \in V$ with $\|u\| = \rho$ one has $I(u) > 0$.*

Then I has a critical value $c > 0$ which is characterized by

$$c = \inf_{\gamma\in\Gamma} \sup_{t\in[0,1]} I[\gamma(t)],$$

where $\Gamma = \{\gamma \in C([0,1],V) : \gamma(0) = 0, \gamma(1) = e\}$.

Proof of Theorem 2.1 Note first that K is a weakly closed convex subset in $H^1_0(\Omega)$ where we equip $H^1_0(\Omega)$ by the following norm:

$$\|u\|^2_{H^1_0(\Omega)} = \int_\Omega (|\nabla u|^2 + \lambda|u|^2)dx.$$

It follows from condition (i) in the theorem that K is compactly embedded in L^p_a. Thus, there exists a constant C such that

$$\|u\|_{H^1_0(\Omega)} \leq \|u\|_V \leq C\|u\|_{H^1_0(\Omega)}, \qquad \forall u \in K. \tag{16}$$

Both the mountain pass geometry and (PS) compactness condition for the function $E_K = \Psi_K - \Phi$ given in (12) follow from the standard arguments together with inequality (16). Here, for the conveience of the reader, we sketch the proof for the (PS) compactness condition and the mountain pass geometry. Suppose that $\{u_n\}$ is a sequence in K such that $E_K(u_n) \to c \in \mathbb{R}$, $\varepsilon_n \to 0$ and

$$\Psi_K(v) - \Psi_K(u_n) + \langle D\Phi(u_n), u_n - v\rangle \geq -\varepsilon_n \|v - u_n\|_V, \qquad \forall v \in V. \tag{17}$$

We must show that $\{u_n\}$ has a convergent subsequence in V. Firstly, we prove that $\{u_n\}$ is bounded in V. Note that since $E_K(u_n) \to c$, then for large values of n we have

$$\frac{1}{2}\|u_n\|^2_{H^1_0(\Omega)} - \frac{1}{p}\int_\Omega a(x)|u|^p dx \le c+1. \tag{18}$$

Note that

$$\langle D\varphi(u_n), u_n\rangle = \int_\Omega a(x)|u_n(x)|^p dx.$$

Thus, by setting $v = ru_n$ in (17) with $r = 1 + 1/p$ we get

$$\frac{(1-r^2)}{2}\|u_n\|^2_{H^1_0(\Omega)} + (r-1)\int_\Omega a(x)|u_n|^p dx \le \varepsilon_n(r-1)\|u_n\|_V. \tag{19}$$

Adding up (19) and (18) yields that

$$\|u_n\|^2_{H^1_0(\Omega)} \le C_0(1 + \|u_n\|_V),$$

for some constant $C_0 > 0$. Therefore, by considering (16), $\{u_n\}$ is bounded in $H^1_0(\Omega)$. Using standard results in Sobolev spaces, after passing to a subsequence if necessary, there exists $\bar{u} \in H^1_0(\Omega)$ such that $u_n \rightharpoonup \bar{u}$ weakly in $H^1_0(\Omega)$ and $u_n \to \bar{u}$ a.e.. Also according to condition (i) in the theorem, from boundedness of $\{u_n\} \subset K$ in $H^1_0(\Omega)$, one can deduce that the strong convergence of u_n to $\bar{u}$ in L^p_a. Now in (17) set $v = \bar{u}$ to get

$$\frac{1}{2}(\|\bar{u}\|^2_{H^1_0(\Omega)} - \|u_n\|^2_{H^1_0(\Omega)}) + \int_\Omega a(x)|u_n|^{p-1}(u_n - \bar{u})dx \ge -\varepsilon_n\|u_n - \bar{u}\|_V. \tag{20}$$

Therefore, it follows from (20) that

$$\frac{1}{2}(\limsup_{n\to\infty}\|u_n\|^2_{H^1_0(\Omega)} - \|\bar{u}\|^2_{H^1}) \le 0.$$

The latter yields that

$$u_n \to \bar{u} \quad \text{strongly in} \quad V$$

as desired. We now verify the mountain pass geometry of the functional E_K. It is clear that $E_K(0) = 0$. Take $e \in K$. It follows that

$$E_K(te) = \frac{t^2}{2}\int_\Omega(|\nabla e|^2 + \lambda|e|^2)dx - \frac{t^p}{p}\int_\Omega a(|x|)|e|^p dx$$

Now, since $p > 2$, for t sufficiently large $E_K(te)$ is negative. Take $u \in K$ with $\|u\|_V = \rho > 0$. We have

$$E_K(u) = \frac{1}{2}\|u\|^2_{H^1} - \frac{1}{p}\int_\Omega a(|x|)|u|^p dx.$$

Note that by (16), there exist positive constant C such that for every $u \in K$ one has

$$\|u\|_{H^1} \le \|u\|_V \le C\|u\|_{H^1}. \tag{21}$$

We also have that

$$\int_\Omega a(|x|)|u|^p dx \le C_0\|u\|^p_V.$$

Therefore

$$E_K(u) \geq \frac{1}{2}\|u\|_{H^1}^2 - \frac{1}{p}\int_\Omega a(|x|)|u|^p dx \geq \frac{1}{2}\|u\|_{H^1}^2 - \frac{C_0}{p}\|u\|_V^p$$
$$\geq \frac{1}{2C^2}\|u\|_V^2 - \frac{C_0}{p}\|u\|_V^p = \frac{1}{2C^2}\rho^2 - \frac{1}{p}\rho^p > 0,$$

provided $\rho > 0$ is small enough, since $p > 2$. If $u \notin K$, then clearly $E_K(u) > 0$. Thus, (MPG) holds for the functional E_K. It now follows from Theorem (2.2) that E_K has a critical point $\bar{u} \in K$, with $E_K(\bar{u}) = c > 0$ where the critical value c is characterized by

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} E_K[\gamma(t)], \tag{22}$$

where

$$\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0 \neq \gamma(1), E_K(\gamma(1)) \leq 0\}.$$

Since $E_K(\bar{u}) > 0$, we have that $\bar{u}$ is non-zero. Since $\bar{u}$ is a critical point of E_K, it follows from Definition 2.1 that

$$\langle D\Phi(\bar{u}), \bar{u} - v\rangle + \Psi_K(v) - \Psi_K(\bar{u}) \geq 0, \qquad \forall v \in V. \tag{23}$$

On the other hand, by (ii), there exists $\bar{v} \in K$ satisfying

$$\begin{cases} -\Delta\bar{v} + \lambda\bar{v} = a(x)|\bar{u}|^{p-2}\bar{u}, & x \in \Omega \\ \bar{v} = 0, & x \in \partial\Omega, \end{cases} \tag{24}$$

in the weak sense. By setting $v = \bar{v}$ in (23) we obtain that

$$\begin{aligned} \frac{1}{2}\int_\Omega |\nabla\bar{v}|^2 + \lambda\bar{v}^2 dx - \frac{1}{2}\int_\Omega |\nabla\bar{u}|^2 + \lambda\bar{u}^2 dx \quad &\geq \quad \int_\Omega a(x)|\bar{u}|^{p-2}\bar{u}(\bar{v} - \bar{u})\, dx \\ &= \quad \int_\Omega \nabla\bar{v}\cdot\nabla(\bar{v} - \bar{u}) + \lambda\bar{v}(\bar{v} - \bar{u})\, dx \end{aligned}$$

where the last equality follows from (24). Therefore,

$$\frac{1}{2}\int_\Omega |\nabla\bar{v} - \nabla\bar{u}|^2\, dx + \frac{\lambda}{2}\int_\Omega |\bar{v} - \bar{u}|^2\, dx \leq 0, \tag{25}$$

which implies that $\bar{u} = \bar{v}$. Taking into account that $\bar{u} = \bar{v}$ in (24) we have that $\bar{u}$ is a weak solution of (11):

$$\begin{cases} -\Delta u + \lambda u = a(x)|u|^{p-2}u, & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases} \tag{26}$$

□

3 Domains of double revolution

In this section we gather some information about the domains of double and higher revolution. We also state and prove useful embedding theorems for these type of domains.
We start by domains of double revolution. Consider writing $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ where $m, n \geq 1$ and $m + n = N$. We define the variables s and t by

$$s := \left\{x_1^2 + \cdots + x_m^2\right\}^{\frac{1}{2}}, \qquad t := \left\{x_{m+1}^2 + \cdots + x_N^2\right\}^{\frac{1}{2}}.$$

We say that $\Omega \subset \mathbb{R}^N$ is a *domain of double revolution* if it is invariant under rotations of the first m variables and also under rotations of the last n variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^N : (s,t) \in U\}$ where U is a domain in $\mathbb{R}^2$ symmetric with respect to the two coordinate axes. In fact,

$$U = \big\{(s,t) \in \mathbb{R}^2 : x = (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, ..., x_N = 0) \in \Omega\big\},$$

is the intersection of Ω with the (x_1, x_{m+1}) plane. Note that U is smooth if and only if Ω is smooth. We denote $\widehat{\Omega}$ to be the intersection of U with the first quadrant of $\mathbb{R}^2$, that is,

$$\widehat{\Omega} = \big\{(s,t) \in U : \ s > 0, \ t > 0\big\}. \tag{27}$$

Using polar coordinates we can write $s = r\cos(\theta)$, $t = r\sin(\theta)$ where $r = |x| = |(s,t)|$ and θ is the usual polar angle in the (s,t) plane.

All domains will be bounded domains in $\mathbb{R}^N$ with smooth boundary unless otherwise stated. To describe the domains in terms of the above polar coordinates we will write

$$\widetilde{\Omega} := \big\{(\theta, r) : (s,t) \in \widehat{\Omega}\big\}. \tag{28}$$

Define

$$H^1_{0,G} := \big\{u \in H^1_0(\Omega) : gu = u \quad \forall g \in G\big\},$$

where $G := O(m) \times O(n)$ where $O(k)$ is the orthogonal group in $\mathbb{R}^k$ and $gu(x) := u(g^{-1}x)$.

In [26] we have considered annular domains and annular domains with monotonicity via the following definition:

Definition 3.1. *We refer to a domain of double revolution in $\mathbb{R}^N$ with $N = m + n$ an annular domain if its associated domain $\widehat{\Omega}$ in the (s,t) plane in $\mathbb{R}^2$ is of the form*

$$\widetilde{\Omega} = \left\{(\theta, r) : g_1(\theta) < r < g_2(\theta), \theta \in \left(0, \frac{\pi}{2}\right)\right\} \tag{29}$$

in polar coordinates. Here $g_i > 0$ is smooth on $[0, \frac{\pi}{2}]$ with $g_i'(0) = g_i'(\frac{\pi}{2}) = 0$ and $g_2(\theta) > g_1(\theta)$ on $[0, \frac{\pi}{2}]$. We call Ω an annular domain with monotonicity *if g_1 is increasing and g_2 is decreasing on $(0, \frac{\pi}{2})$.*

To distinguish these domains from the new ones we will refer to these as $\frac{\pi}{2}$ annular domains with and without monotonicity. We proved the following imbeddings:

Theorem A. *[26] Let Ω denote a $\frac{\pi}{2}$ annular domain in $\mathbb{R}^N$.*

1. *(Imbedding without monotonicity) Suppose Ω has no monotonicity and*

$$1 \le p < \min\left\{\frac{2(n+1)}{n-1}, \frac{2(m+1)}{m-1}\right\}.$$

Then $H^1_{0,G}(\Omega) \subset\subset L^p(\Omega)$ with the obvious interpretation in the case of $m = n = 1$.

2. *(Imbedding with monotonicity) Suppose Ω has monotonicity, $n \le m$ and*

$$1 \le p < \frac{2(n+1)}{n-1} = \max\left\{\frac{2(n+1)}{n-1}, \frac{2(m+1)}{m-1}\right\}.$$

Then $K_{-,\frac{\pi}{2}} \subset\subset L^p(\Omega)$ with the obvious interpretation if $n = 1$ where

$$K_{-,\frac{\pi}{2}} = \left\{0 \le u \in H^1_{0,G}(\Omega) : u_\theta \le 0 \ a.e. \ in \ \widetilde{\Omega}\right\}.$$

Remark 3.2. *1. The above imbedding makes sense with a bit of heuristics. Consider an annular domain in $\mathbb{R}^N$ with $N = m + n$ and we suppose $n \le m$. Suppose we are given a sequence of functions $0 \le u_k \in H^1_{0,G}(\Omega)$. If the functions concentrate near $t = 0$ then the problem looks like a problem in dimension $n+1$ (ie. the t variable has dimension n and the s variable has dimension 1 since we are away from $s = 0$) and hence the critical Sobolev exponent $\frac{2(n+1)}{(n+1)-2}$ should play a role. The functions can also concentrate near $s = 0$ and then the relevant exponent is $\frac{2(m+1)}{(m+1)-2}$. The functions can also concentrate in other regions but they are of lower dimension and hence doesn't limit the imbedding. This suggests part 1 of Theorem A.*

2. To see part 2 of Theorem A we note that we now have monotonicity in θ and hence the functions only have the option to concentrate on $\theta = 0$ or on the s axis and hence this gives the improved result.

Before going into more details we give some more background on domains of double revolution.

Assume Ω is a domain of double revolution and v is a function defined on Ω that just depends on (s,t), then one has

$$\int_\Omega v(x)dx = c(m,n)\int_{\widehat{\Omega}} v(s,t)s^{m-1}t^{n-1}dsdt,$$

where $c(m,n)$ is a positive constant depending on n and m. Note that strictly speaking we are abusing notation here by using the same name; and we will continuously do this in this article. Given a function v defined on Ω we will write $v = v(s,t)$ to indicate that the function has this symmetry.

To solve equations on domains of double revolution one needs to relate the equation to a new one on $\widehat{\Omega}$ defined in (27). Suppose Ω is a domain of double revolution and f has is function defined on Ω with the same symmetry (ie. $gf(x) = f(g^{-1}x)$ all $g \in G$). Suppose that $u(x)$ solves

$$\begin{cases} -\Delta u(x) = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{30}$$

Then $u = u(s,t)$ and u solves

$$-u_{ss} - u_{tt} - \frac{(m-1)u_s}{s} - \frac{(n-1)u_t}{t} = f(s,t) \text{ in } \widehat{\Omega}, \tag{31}$$

with $u = 0$ on $(s,t) \in \partial\widehat{\Omega}\backslash(\{s = 0\} \cup \{t = 0\})$. If u is sufficiently smooth then $u_s = 0$ on $\partial\widehat{\Omega} \cap \{s = 0\}$ and $u_t = 0$ on $\partial\widehat{\Omega} \cap \{t = 0\}$ after considering the symmetry properties of u.

One can easily refine the notion of the domain of double revolution to domains of m revolution.

Domains of m revolution. Consider writing $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}$ where $n_1 + \cdots + n_m = N$ and $n_1, ..., n_m \ge 1$. We say that $\Omega \subset \mathbb{R}^N$ is a *domain of m revolution* if it is invariant under rotations of the first n_1 variables, the next n_2 variables, ..., and finally in the last n_m variables. We define the variables t_i via

$$t_1^2 := x_1^2 + \cdots + x_{n_1}^2, \quad t_2^2 := x_{n_1+1}^2 + \cdots + x_{n_1+n_2}^2,$$

and similar for t_i for $3 \le i < m$. Finally we define

$$t_m^2 := \sum_{k=n_1+n_2+\cdots+n_{m-1}+1}^{N} x_k^2.$$

We now define

$$U = \Big\{ t \in \mathbb{R}^m ; x = (x_1, ..., x_N) \in \Omega, \text{ where } x_1 = t_1,\ x_{n_1+n_2+\cdots+n_{k-1}+1} = t_k \text{ for } 2 \leq k \leq m, \text{ and } x_i = 0 \text{ for } i \neq 1, n_1+1, n_1+n_2+1, ..., n_1+n_2+...+n_{m-1}+1 \Big\}.$$

We define $\widehat{\Omega} \subset \mathbb{R}^m$ to be the intersection of U with the first sector of $\mathbb{R}^m$. We now define the appropriate measure

$$d\mu_m(t) = d\mu_m^{(n_1,...,n_m)}(t_1, ..., t_m) = \prod_{k=1}^{m} t_k^{n_k-1} dt_k.$$

Given any function v defined in Ω, that depends only on the radial variables $t_1, t_2, .., t_m$ one has

$$\int_\Omega v(x)dx = c(n_1, ..., n_m) \int_{\widehat{\Omega}} v(t) d\mu_m(t),$$

where $c(n_1, ..., n_m)$ just depends on $n_1, ..., n_m$. Given that $\Omega \subset \mathbb{R}^N$ is a domain of m revolution with $\sum_{i=1}^m n_i = N$, let

$$G := O(n_1) \times O(n_2) \times ... \times O(n_m),$$

where $O(n_i)$ is the orthogonal group in $\mathbb{R}^{n_i}$ and consider

$$H^1_{0,G} := \left\{ u \in H^1_0(\Omega) : gu = u \quad \forall g \in G \right\},$$

where $gu(x) := u(g^{-1}x)$. If $u \in H^1_{0,G}$ then u has symmetry compatibility with Ω, ie. $u(x)$ depends on just $t_1, ..., t_m$ and we write this as $u(x) = u(t_1, ..., t_m)$ where $(t_1, ..., t_m) \in \widehat{\Omega}$. We have the following embedding result for the domains of m revolution.

Theorem 3.1. *Let Ω denote a bounded domain of m revolution in $\mathbb{R}^N$ with $N = n_1 + ... + n_m$ and $n_i \geq 1$ such that $0 \notin \bar{\Omega}$. Assume that*

$$1 \leq p < \min \left\{ \frac{2(N - n_i + 1)}{N - n_i - 1};\ i = 1, ..., m \right\}.$$

Then $H^1_{0,G}(\Omega) \subset\subset L^p(\Omega)$ with the obvious interpretation in the case of $N - n_i = 1$.

Proof. Assume that $x = (y_1, ..., y_m) \in \mathbb{R}^N = \Pi_{i=1}^m \mathbb{R}^{n_i}$. Let R_1 and R_2 be such that $0 < R_1 < |x| < R_2$ for all $x \in \Omega$. Choose δ small enough such that $\sqrt{m}\delta < R_1$. It then follows that for each $x = (y_1, ..., y_m) \in \Omega$ we have that $|y_i| \geq \delta$ for at least one $i \in \{1, ..., m\}$. Therefore,

$$\begin{aligned} \int_\Omega |u|^p\, dx \leq \Sigma_{i=1}^m \int_{\Omega,\, |y_i| \geq \delta} |u|^p\, dx \quad &\leq \quad \Sigma_{i=1}^m c_i \int_{\Omega,\, |r_i| \geq \delta} r_i^{n_i-1} |u(y_1, ..., y_{i-1}, r_i, y_{i+1}, .., y_m)|^p\, dy_1 ...dr_i...dy_m \\ &\leq \quad \Sigma_{i=1}^m c_i R_2^{n_i-1} \int_{\Omega,\, |r_i| \geq \delta} |u(y_1, ..., y_{i-1}, r_i, y_{i+1}, .., y_m)|^p\, dy_1 ...dr_i...dy_m \end{aligned}$$

for appropriate constants c_i. Morovere, for

$$1 < p < \frac{2(N - n_i + 1)}{N - n_i - 1},$$

we have that

$$\int_{\Omega,\, |r_i| \geq \delta} |u(y_1, ..., y_{i-1}, r_i, y_{i+1}, .., y_m)|^p\, dy_1 ...dr_i...dy_m$$

is being controlled by the $H_1(\Omega,\,|r_i|\ge\delta)$. On the other hand

$$\int_{\Omega,\,|r_i|\ge\delta}\left(|\nabla u(y_1,...,y_{i-1},n_i,y_{i+1},..,y_m)|^2+|u(y_1,...,y_{i-1},r_i,y_{i+1},..,y_m)|^2\right)dy_1\,...dr_i...dy_m \le$$

$$\delta^{-n_i+1}\int_{\Omega,\,|r_i|\ge\delta} r_i^{n_i-1}\left(|\nabla u(y_1,...,y_{i-1},r_i,y_{i+1},..,y_m)|^2+|u(y_1,...,y_{i-1},r_i,y_{i+1},..,y_m)|^2\right)dy_1\,...dr_i...dy_m \le$$

$$C_i\delta^{-n_i+1}\int_{\Omega}\left(|\nabla u(y_1,...,y_{i-1},y_i,y_{i+1},..,y_m)|^2+|u(y_1,...,y_{i-1},y_i,y_{i+1},..,y_m)|^2\right)dy_1\,...dy_i...dy_m =$$

$$C_i\delta^{-n_i+1}\|u\|^2_{H_1(\Omega)},$$

for appropriate constants C_i. This completes the proof. □

4 Supercritical elliptic problems on domains of double revolution

In this section we examine the equation

$$\left\{\begin{array}{ll} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{array}\right. \tag{32}$$

where Ω is a domain of double revolution in $\mathbb{R}^N=\mathbb{R}^n\times\mathbb{R}^n$. Note when $m=n$, Theorem A does not show any improvements in compactness when using monotonicity. In this case the equation has a certain invariance across $\theta=\frac{\pi}{4}$ and this suggests one examine domains with a certain invariance also. This brings us to a first type of new domains.

Definition 4.1. *We will call a domain of double revolution in $\mathbb{R}^N$ a $\frac{\pi}{4}$-*annular domain with monotonicity *provided the domain is an annular domain via Definition 3.1 (ie. $g_i>0$ is smooth on $[0,\frac{\pi}{2}]$ with $g_i'(0)=g_i'(\frac{\pi}{2})=0$ and $g_2(\theta)>g_1(\theta)$ on $[0,\frac{\pi}{2}]$) and g_1 is increasing and g_2 is decreasing on $(0,\frac{\pi}{4})$ and both g_1,g_2 are even across $\theta=\frac{\pi}{4}$. For these new domains we define a suitable subset of $\widetilde{\Omega}$ given by*

$$\widetilde{\Omega}_0=\left\{(\theta,r): g_1(\theta)<r<g_2(\theta), 0<\theta<\frac{\pi}{4}\right\}. \tag{33}$$

We now are in a position to define the class of functions we work on in this setting.

1. (K_-) In the case of Ω a $\frac{\pi}{4}$-annular domain with monotonicity (see Definition 4.1) we define K_- to be the set of nonnegative functions $u\in H^1_{0,G}(\Omega)$ with $u_\theta\le 0$ in $\widetilde{\Omega}_0$ and which are even across $\theta=\frac{\pi}{4}$.

2. (K_+) In the case of Ω an annulus we define K_+ to be the set of nonnegative functions $u\in H^1_{0,G}(\Omega)$ with $u_\theta\ge 0$ in $\widetilde{\Omega}_0$ and which are even across $\theta=\frac{\pi}{4}$.

Note K_- is defined for an annulus and a more general annular domain with the added assumptions where as we only define K_+ for an annulus. Our approach utilizing K_+ will fail on a more general annular domain. The imbeddings we prove regarding K_- are essentially the same as Theorem A. For K_+ one expects to get more. Before we state our main theorem for this section we need to define a quantity that will be relevant to showing the ground states on radial domains are nonradial and this quantity will be relevant for the equaitons that follow in later sections also. Indeed, we define

$$\beta_0(\Omega):=\inf_{u\in H^1_0(\Omega)}\frac{\int_\Omega|\nabla u|^2\,dx}{\int_\Omega\frac{u^2}{|x|^2}\,dx}. \tag{34}$$

Note this quantity is just the best constant in the classical Hardy inequality. So if $0 \in \Omega$ or Ω is an exterior domain then $\beta_0(\Omega) = \frac{(N-2)^2}{4}$.

Theorem 4.1. *Let Ω be a bounded domain in $\mathbb{R}^N$ with $N = 2n$.*

1. *Suppose Ω is a bounded $\frac{\pi}{4}$-annular domain with monotonicity, $a = a(s,t)$ is positive and sufficiently smooth and $a_\theta \le 0$ in $\widetilde{\Omega}_0$. Then for all*

$$2 < p < \frac{2N+4}{N-2},$$

there is a positive classical K_- ground state solution u of (32). Note this case includes the case of Ω an annulus.

2. *Suppose Ω is an annulus with $a = a(s,t)$ positive and sufficiently smooth and $a_\theta \ge 0$ in $\widetilde{\Omega}_0$.*

   2-a Then for all $2 < p < \infty$ there is a positive classical K_+ ground state solution u of (32).

   2-b Moreover, if a is a radial function then for

$$\frac{4(N+2)}{\beta_0(\Omega)} < p < \infty,$$

   the ground state solution u in 2-a in nonradial.

We shall make use of Theorem 2.1 to prove the above result. In that regard, we shall need to verify two conditions in Theorem 2.1, namely, the compact embedding and the point wise invariance property.

Proposition 4.1. *($\frac{\pi}{4}$- annular domain imbeddings) Suppose $n = m = \frac{N}{2}$.*

1. *(K_- imbedding) Suppose Ω is $\frac{\pi}{4}$-annular domain with monotonicity and*

$$1 \le p < \frac{4(N+1)}{N-2}.$$

   Then $K_- \subset\subset L^p(\Omega)$.

2. *(K_+ imbedding) Suppose Ω is an annulus in $\mathbb{R}^N$ and $1 \le p < \infty$. Then $K_+ \subset\subset L^p(\Omega)$.*

Proof. Part 1: The proof used in the proof of Theorem A carries over to this case.
Part 2: If we take $u \in K_+$ note that the function is largest at $\theta = \frac{\pi}{4}$. So note the problems appears to be a genuine two dimensional problem near $\theta = \frac{\pi}{4}$ and hence we expect to have imbeddings for all p, see Remark 3.2 for related comments. For concreteness we work on the annulus centered at the origin with inner radius 1 and outer radius 2.

Then note for $0 \le u \in H^1_{0,G}(\Omega)$ (which are also even about $\theta = \frac{\pi}{4}$ but may not have any monotonicity) we have

$$\int_\Omega u(x)^p dx = \int_1^2 \int_0^{\frac{\pi}{4}} u(r,\theta)^p r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr,$$

and

$$\int_\Omega |\nabla u(x)|^2 dx = \int_1^2 \int_0^{\frac{\pi}{4}} \left\{ u_r^2 + \frac{u_\theta^2}{r^2} \right\} r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr.$$

For any $1 \leq p < \infty$ there is some $C_p > 0$ (independent of u as above) such that

$$\left\{ \int_1^2 \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} u(r,\theta)^p r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr \right\}^{\frac{2}{p}}, \tag{35}$$

is bounded above by

$$C_p \int_1^2 \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} \left\{ u_r^2 + \frac{u_\theta^2}{r^2} \right\} r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr.$$

The two important points are that the integrals are over $1 < r < 2$ and $\frac{\pi}{8} < \theta < \frac{\pi}{4}$. Note on this range of θ and r the measure $d\mu(r,\theta) = r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr$ is essentially two dimensional, ie. comparable to $d\theta dr$. This allows one to use the two dimensional Sobolev imbedding. To see this more rigously one can consider working on $(r,\theta) \in (1,2) \times (\frac{\pi}{8}, \frac{\pi}{4})$ and hence we can consider the Sobolev imbeddings in the product space. Let $u \in K_+$ and then note that

$$\begin{aligned} \int_1^2 \int_0^{\frac{\pi}{8}} u(r,\theta)^p r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr &\leq \int_1^2 \int_0^{\frac{\pi}{8}} u(r, \theta + \frac{\pi}{8})^p r^{2n-1} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr \\ &\leq \int_1^2 \int_{\frac{\pi}{8}}^{\frac{\pi}{4}} u(r,\hat{\theta})^p r^{2n-1} d\hat{\theta} dr \end{aligned}$$

where in the first line we used the monotonicity of u. Note this final quantity is bounded above by the $\frac{p}{2}$ power of (35). We can now combine the results which completes the proof of part 2. □

The following theorem develops pointwise invariance property (see Theorem 2.1 part (ii)) which is related to the linear problem

$$\begin{cases} -\Delta v = a(x) u^{p-1} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{36}$$

Proposition 4.2. *(Pointwise invariance property; case $m = n = \frac{N}{2}$). Suppose a is nonnegative with $a = a(s,t)$, $a_\theta = s a_t - t a_s$ is bounded and a is even across $\theta = \frac{\pi}{4}$.*

1. *Suppose Ω is $\frac{\pi}{4}$-annular domain with monotonicity and $a_\theta \leq 0$ in $\widetilde{\Omega}_0$. If $u \in K_-$ and v satisfies (36) then $v \in K_-$.*

2. *Suppose Ω is an annulus and $a_\theta \geq 0$ in $\widetilde{\Omega}_0$. If $u \in K_+$ and v satisfies (36) then $v \in K_+$.*

Proof. Much of the proof won't depend on which case we are in. Additionally we have $m = n$ but for the time being we won't indicate this since many of these computations will be useful in later cases where they are not equal. Let $u \in K_\pm$ and for k large consider $u_k(x) = \min\{u(x), k\}$ and note that $u_k \in K_\pm$. Let v^k denote a solution of

$$\begin{cases} -\Delta v = a(x) u_k^{p-1} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{37}$$

By elliptic regularity we have $v^k \in H^1_{0,G}(\Omega) \cap C^{1,\alpha}(\overline{\Omega})$ for any $0 < \alpha < 1$. In terms of (s,t) we see that v^k satisfies

$$-v^k_{ss} - v^k_{tt} - \frac{(m-1)v^k_s}{s} - \frac{(n-1)v^k_t}{t} = a u_k^{p-1} \quad \text{in } \widehat{\Omega}, \tag{38}$$

with $v^k = 0$ on $(s,t) \in \partial\widehat{\Omega} \backslash (\{s=0\} \cup \{t=0\})$. Since v^k is sufficiently smooth then $v_s^k = 0$ on $\partial\widehat{\Omega} \cap \{s=0\}$ and $v_t^k = 0$ on $\partial\widehat{\Omega} \cap \{t=0\}$ after considering the symmetry properties of v^k (see [26] for details). We now want to show that v^k has the added symmetry across the line $t = s$. Here there are a few ways to argue. We can directly use the (s,t) coordinates or we can switch to polar coordinates, we will use the second approach.

A computation shows that

$$\frac{(m-1)v_s^k}{s} + \frac{(n-1)v_t^k}{t} = \frac{(N-2)v_r^k}{r} + \frac{v_\theta^k}{r^2}\left\{\frac{n-1}{\tan(\theta)} - (m-1)\tan(\theta)\right\}$$

if we write the equation in terms of polar coordinates (recall we have $s = r\cos(\theta), t = r\sin(\theta)$). Writing out (38) in polar coordinates gives

$$-v_{rr}^k - \frac{(N-1)v_r^k}{r} - \frac{v_{\theta\theta}^k}{r^2} + \frac{v_\theta^k}{r^2}h(\theta) = au_k^{p-1}, \text{ in } \widetilde{\Omega}, \tag{39}$$

with $v^k = 0$ on $\partial\widetilde{\Omega} \backslash (\Gamma_L \cup \Gamma_R)$ where Γ_L (respectively Γ_R) corresponds to the portion of $\partial\widetilde{\Omega}_0$ given by $\{\theta = 0\}$ (respectively $\{\theta = \frac{\pi}{2}\}$) and where $v_\theta^k = 0$ on $\Gamma_L \cup \Gamma_R$ and where

$$h(\theta) = (m-1)\tan(\theta) - \frac{(n-1)}{\tan(\theta)}. \tag{40}$$

We now show that v^k is even across $\theta = \frac{\pi}{4}$; so we set $\widehat{v}(r,\theta) = v^k(r, \frac{\pi}{2} - \theta)$ and we want to show that $\widehat{v} = v^k$ in $\widetilde{\Omega}$. Because of the smoothness of v^k we have $\partial_\theta v^k = 0$ at $\theta = 0, \frac{\pi}{2}$ and hence we have the same for $\widehat{v}$. Also note that since $m = n$ we have h is odd across $\theta = \frac{\pi}{4}$, ie.

$$h(\theta) = -h(\frac{\pi}{2} - \theta)$$

for $0 < \theta < \frac{\pi}{2}$. Note the right hand side of (39) is even across $\theta = \frac{\pi}{4}$. From this we see $\widehat{v}$ satisfies (39) with the same boundary conditions and hence by uniqueness of solution we have $\widehat{v} = v^k$ in $\widetilde{\Omega}$. Now since v^k is even across $\theta = \frac{\pi}{4}$ and v^k is sufficiently smooth we have $v_\theta^k = 0$ on $\theta = \frac{\pi}{4}$.

Monotonicity. Let $w = v_\theta^k$ and then note that if we take a derivative in θ of the equation for v^k we arrive at

$$-w_{rr} - \frac{(N-1)w_r}{r} - \frac{w_{\theta\theta}}{r^2} + \frac{w_\theta}{r^2}h(\theta) + \frac{w}{r^2}h'(\theta) = \partial_\theta\left\{au_k^{p-1}\right\}, \text{ in } \widetilde{\Omega}, \tag{41}$$

and in particular the equation is satisfied in $\widetilde{\Omega}_0$ with $w = 0$ on the portion of $\partial\widetilde{\Omega}_0$ corresponding to $\theta = 0, \frac{\pi}{4}$. A computation shows that if write the left hand side of (41) in terms of x we arrive at

$$-\Delta w(x) + \frac{(n-1)|x|^2 w(x)}{(x_1^2 + \cdots + x_m^2)(x_{m+1}^2 + \cdots + x_N^2)},$$

which, at least formally, satisfies a maximum principle.

We now separate the cases of $u \in K_-$ and $u \in K_+$. Suppose $u \in K_+$ and Ω an annulus. Then $w = 0$ on the curved portions of $\widetilde{\Omega}_0$ since $v^k = 0$ on these portions of the boundary. Also note that the right hand side of (41) is nonnegative and assuming we can apply the maximum principle we arrive at $w \geq 0$ in $\widetilde{\Omega}_0$.

We now suppose $u \in K_-$. Then we have the right hand side of (41) is nonpositive. Since $v^k \geq 0$ in $\widetilde{\Omega}_0$ and noting the monotnicity of g_1 and g_2 we see that $w = v^k_\theta \leq 0$ on the curved portions of $\partial\widetilde{\Omega}_0$ and again if we can apply the maximum principle we arrive at $w \leq 0$ in $\widetilde{\Omega}_0$.

To make these maximum principle arguments used above rigorous we use the idea of [12] (see also [26]). Consider the case of $u \in K_-$. Let $\varepsilon > 0$ be small and consider $\psi := (w - \varepsilon)_+$. By the smoothness properties of v^k and noting the boundary values of v^k we have $w = 0$ near $\theta = 0$ and $\theta = \frac{\pi}{4}$. Using ψ as a test function on a suitable weak notion of a solution of (41) one will arrive at $\psi = 0$ and sine $\varepsilon > 0$ is arbitrary we have $w \leq 0$.

Sending $k \to \infty$. We now get bounds on v^k which allow us to pass to the limit in k. We assume that $u \in K_-$ and v^k as above. Then testing the weak formulation for v^k on v^k gives

$$\begin{aligned}
\int_\Omega |\nabla v^k|^2 dx &= \int_\Omega a u_k^{p-1} v^k dx \\
&\leq C \|u_k^{p-1}\|_{L^{p'}} \|v^k\|_{L^p} \\
&\leq C_0 \|u_k\|^{p-1}_{L^{p'(p-1)}} \|\nabla v^k\|_{L^2} \\
&= C_0 \|u_k\|^{p-1}_{L^p} \|\nabla v^k\|_{L^2}
\end{aligned}$$

where the second last inequality follows by part 1 of by Proposition 4.1 after noting the restriction on p and the final equality follows since $p'(p-1) = p$. Using the imbedding again we arrive at

$$\|\nabla v^k\|_{L^2} \leq C_0 \|u_k\|^{p-1}_{L^p} \leq C_1 \|\nabla u_k\|^{p-1}_{L^2},$$

since $u_k \in K_-$, and now note this quantity on the right is bounded independently of k and hence v^k is bounded in $H^1_{0,G}(\Omega)$ and after passing to a subsequence we can assume that there is some $v \in H^1_{0,G}(\Omega)$ such that $v^k \rightharpoonup v$ in $H^1_{0,G}(\Omega)$ and its clear that v is an $H^1_{0,G}(\Omega)$ solution of (36). Also note that $u_k^{p-1} \to u^{p-1}$ in $L^{p'}(\Omega)$ and hence by passing to another subsequence we have $v^k \to v$ in $W^{2,p'}(\Omega)$ and hence we can assume $\nabla v^k \to \nabla v$ in $L^{p'}(\Omega)$ and a.e. in Ω. We now suppose that $0 \leq \psi \in C^\infty_c(\widetilde{\Omega}_0)$ and note that we have

$$0 \geq \int_{\widetilde{\Omega}_0} v^k_\theta \psi dr d\theta = -\int_{\widetilde{\Omega}_0} v^k \psi_\theta dr d\theta,$$

and noting that $v^k \to v$ in $L^2_{loc}(\widetilde{\Omega}_0, dr d\theta)$ (recall we are away from the origin in this problem and the measures only have issues on $\theta = 0, \frac{\pi}{2}$)) and hence we can pass to the limit here to see that $0 \geq \int_{\widetilde{\Omega}_0} v \psi_\theta dr d\theta$ but this is sufficient to see that $v_\theta \geq 0$ a.e. in $\widetilde{\Omega}_0$.

The case of $u \in K_+$ has a similar proof and we skip the details. □

Proof of Theorem 4.1. We are going to use Theorem 2.1 for the proof. Note that conditions (i) and (ii) in Theorem 2.1 follows from Propositions 4.1 and 4.2 respectively. This proves the existence of a weak solution of (32) for both cases (*1*) and (*2-a*). It also follows from Theorem 8.1 that for

$$\frac{4(N+2)}{\beta_0(\Omega)} < p < \infty,$$

the ground state solution u in *2-a* is non-radial.

Regularity of the solution. We will prove the case of part 1, the case of part 2 is easier since one doesn't need an iteration. Let $q := \frac{2(n+1)}{n-1}$ and take $t_0 = 1$ and

$$t_{k+1} := \frac{qt_k}{2} - \frac{p-2}{2},$$

where $1 < p < q$. Then by examining the cobweb we see that $t_k \to \infty$.

We now prove the following inductive step. If $k \geq 0$ and $u^{t_k} \in K_-$ then $u^{t_{k+1}} \in K_-$. Assuming this is true for a moment then note we see that since $u^{t_0} = u \in K_-$ we can iterate to see $u \in L^T(\Omega)$ for all $T < \infty$ and hence we see that u is $C^{1,\delta}(\overline{\Omega})$ and then we can proceed with the Schauder regularity theory and the exact smoothness of u will depend on the smoothness of a. Assuming a at least Hölder continuous we have u is a classical solution.

We now prove the iteration step. Suppose $u^{t_k} \in K_-$ for some $k \geq 0$ and for i a large integer define

$$\varphi(x) = \begin{cases} u(x)^{2t_k - 1} & \text{if } u(x) < i, \\ i^{2t_k - 1} & \text{if } u(x) \geq i. \end{cases} \tag{42}$$

Note that $\varphi \in K_-$. We can test (32) on φ to arrive at (here $\Omega_i := \{x \in \Omega : u(x) < i\}$)

$$\begin{aligned} \frac{(2t_{k+1}-1)}{t_{k+1}^2} \int_{\Omega_i} |\nabla u^{t_{k+1}}|^2 dx &= \int_\Omega a u^{p-1} \varphi dx \\ &= \int_{\Omega_i} a u^{p+2t_{k+1}-2} dx + \varepsilon_{k,i} \\ &= \int_{\Omega_i} a \left(u^{t_k}\right)^{\frac{p+2t_{k+1}-2}{t_k}} dx + \varepsilon_{k,i} \\ &= \int_{\Omega_i} a \left(u^{t_k}\right)^q dx + \varepsilon_{k,i} \\ &\leq \int_\Omega a \left(u^{t_k}\right)^q dx + \varepsilon_{k,i}, \end{aligned}$$

where

$$\varepsilon_{k,i} := i^{2t_{k+1}-1} \int_{\Omega \backslash \Omega_i} a u^{p-1} dx = i^{qt_k - (p-1)} \int_{\Omega \backslash \Omega_i} a u^{p-1} dx.$$

First note since $u^{t_k} \in K_-$ then we see the $u^{t_k} \subset L^q(\Omega)$ by the imbedding and hence the integral on the right is finite. Set $C_k = \int_\Omega u^{t_k q} dx$ and note we have

$$i^{t_k q} |\Omega \backslash \Omega_i| \leq C_k,$$

for all large i. Put $\delta_{k,i} := \int_{\Omega \backslash \Omega_i} u^{t_k q} dx$ and note $\delta_{k,i} \to 0$ as $i \to \infty$. Let $(p-1)\tau = t_k q$ and then note

$$\begin{aligned} \frac{\varepsilon_{k,i}}{i^{qt_k - (p-1)}} &= \int_{\Omega \backslash \Omega_i} a u^{p-1} dx \\ &\leq C_a \left(\int_{\Omega \backslash \Omega_i} u^{t_k q} dx \right)^{\frac{1}{\tau}} |\Omega \backslash \Omega_i|^{\frac{1}{\tau'}} \end{aligned}$$

so we have

$$\frac{\varepsilon_{k,i}^{\tau'}}{i^{(qt_k-(p-1))\tau'}} \quad \leq \quad C_a^{\tau'}\delta_{k,i}^{\frac{\tau'}{\tau}}\frac{C_k}{i^{t_k q}}$$

which gives us

$$\varepsilon_{k,i}^{\tau'} \leq C_a^{\tau'}\delta_{k,i}^{\frac{\tau'}{\tau}}C_k \to 0,$$

as $i \to \infty$. From this we see that

$$\frac{(2t_{k+1}-1)}{t_{k+1}^2}\int_\Omega |\nabla u^{t_{k+1}}|^2 dx \leq \int_\Omega a(u^{t_k})^q dx < \infty,$$

and hence we see that $u^{t_k+1} \in H_{0,G}^1(\Omega)$ and its clear the monotonicity and symmetry is sufficient that $u^{t_{k+1}} \in K_-$. □

5 Hénon equation on B_1 in even dimensions

In this section we examine the Hénon equation given by

$$\begin{cases} -\Delta u = |x|^\alpha u^{p-1} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{43}$$

where B_1 is the unit ball in $\mathbb{R}^N$ centered at the origin and $N \geq 3$ and $\alpha > 0$. Our interest is in obtaining positive classical nonradial solutions in the supercritical case

$$\frac{2N}{N-2} < p < \frac{2N+2\alpha}{N-2},$$

via our variational approach. In the radial case the weight improves compactness of the Sobolev imbedding to $H_{0,rad}^1(B_1) \subset\subset L^p(B_1, |x|^\alpha dx)$ to $1 \leq p < \frac{2N+2\alpha}{N-2}$, see [52] and this allows one to obtain a positive radial solution for this range of p. The first work to obtain a nonradial solution was in [62] in the subcritical case. This was later extended to other values of p in [4, 34, 44, 55, 56]. Many of these works used bifurcation approaches to show the existence of nonradial solutions.

We now define K_+ in essentially the same way we did on the annulus;

$$K_+ = \left\{0 \leq u \in H_{0,G}^1(B_1) : u \text{ is even in } \theta \text{ across } \theta = \tfrac{\pi}{4} \text{ with } u_\theta \geq 0 \text{ for } 0 < r < 1 \text{ and } 0 < \theta < \tfrac{\pi}{4} \right\},$$

and we define $\widetilde{\Omega}$ and $\widetilde{\Omega}_0$ in the obvious way after considering the definitions in Section 3. We will not consider working on K_- here even though it would give a different type of solution as compared to K_+, but one would need to further restrict the upper bound on p and so we chose not to include this.

Here is our main theorem in this section.

Theorem 5.1. *(K_+ solutions for Hénon equation). Let B_1 is the unit ball in $\mathbb{R}^N$ centered at the origin, $N \geq 4$ is even and $\alpha > 0$. The following assertions hold:*

1. *Suppose $2 < p < \frac{2N+2\alpha}{N-2}$. Then there is a positive classical K_+ ground state solution u of (43).*

2. Suppose

$$\frac{16(N+2)}{(N-2)^2}+2<p<\frac{2N+2\alpha}{N-2}.$$

Then the positive classical K_+ ground state solution u of (43) is nonradial.

Remark 5.1. *Note all these results can immediately give results regarding fast decay solutions of related problems on exterior domains after applying a Kelvin transform.*

We shall need some preliminaries before proving this theorem.

Proposition 5.1. *(Imbedding iteration) Let Ω denote a bounded domain of double revolution in $\mathbb{R}^N=\mathbb{R}^{m+n}$ (here m and n need not be equal). For all integers $k\geq 0$ there is some $C_k\geq 0$ such that for all $0\leq\varphi\in H^1_{0,G}(\Omega)$ with $\|\nabla\varphi\|_{L^2(\Omega)}=1$ we have*

$$\int_{\widehat{\Omega}}(\varphi(s,t))^{2^*+2k}s^{(k+1)(m-1)}t^{(k+1)(n-1)}dsdt\leq C_k, \tag{44}$$

where $2^*=2^*_N=\frac{2N}{N-2}$.

Proof. Take $0\leq u\in H^1_{0,G}(\Omega)$ (and say Lipschitz) and take $\beta_i>0$. By extending u to the full first quadrant by extending it to be zero outside of $\widehat{\Omega}$ we have

$$u(s,t)\leq\int_s^\infty|\nabla_{s,t}u(\tau_1,t)|d\tau_1,$$

$$u(s,t)\leq\int_t^\infty|\nabla_{s,t}u(s,\tau_2)|d\tau_2,$$

and hence we have

$$u(s,t)^2\leq\int_s^\infty|\nabla_{s,t}u(\tau_1,t)|d\tau_1\int_t^\infty|\nabla_{s,t}u(s,\tau_2)|d\tau_2,$$

and we now multiply by sides by $s^{2\beta_1}t^{2\beta_2}$ where $\beta_i>0$ and integrate over $\widehat{\Omega}$ we arrive at

$$\int_{\widehat{\Omega}}u(s,t)^2s^{2\beta_1}t^{2\beta_2}dsdt\leq\left(\int_{\widehat{\Omega}}|\nabla_{s,t}u(s,t)|s^{\beta_1}t^{\beta_2}dsdt\right)^2. \tag{45}$$

We now suppose $0\leq\varphi\in H^1_{0,G}(\Omega)$ is smooth and with the gradient assumption as in the hypothesis and we put $u=\varphi^\gamma$ into (45) where $\gamma\geq 1$. Then we arrive at

$$\begin{aligned}\int_{\widehat{\Omega}}\varphi^{2\gamma}s^{2\beta_1}t^{2\beta_2}dsdt &\leq \gamma^2\left(\int_{\widehat{\Omega}}\left\{|\nabla_{s,t}\varphi|s^{\frac{m-1}{2}}t^{\frac{n-1}{2}}\right\}\left\{\varphi^{\gamma-1}s^{\beta_1-\frac{m-1}{2}}t^{\beta_2-\frac{n-1}{2}}\right\}dsdt\right)^2\\ &\leq \gamma^2\|\nabla\varphi\|^2_{L^2(\Omega)}\int_{\widehat{\Omega}}\varphi^{2(\gamma-1)}s^{2\beta_1-(m-1)}t^{2\beta_2-(n-1)}dsdt,\end{aligned}$$

where we performed the Cauchy–Schwarz inequality and recall $\|\nabla\varphi\|_{L^2}=1$. We will now use this inequality to perform an iteration in γ and β_i. For $k\geq 0$ define

$$\gamma_k=\frac{2^*}{2}+k,\quad \beta_1^k=\frac{(k+1)(m-1)}{2},\quad \beta_2^k=\frac{(k+1)(n-1)}{2}.$$

Now suppose for $k\geq 1$ we have

$$\int_{\widehat{\Omega}}\varphi^{2\gamma_{k-1}}s^{2\beta_1^{k-1}}t^{2\beta_2^{k-1}}dsdt=C_k,$$

then by putting $\gamma = \gamma_k$ and $\beta_1 = \beta_1^k, \beta_2 = \beta_2^k$ into (45) we arrive at

$$\begin{aligned}\int_{\widehat{\Omega}} \varphi^{2\gamma_k} s^{2\beta_1^k} t^{2\beta_2^k} dsdt &\leq \gamma_k^2 \int_{\widehat{\Omega}} \varphi^{2(\gamma_k - 1)} s^{2\beta_1^k - (m-1)} t^{2\beta_2^k - (n-1)} dsdt \\ &= \gamma_k^2 \int_{\widehat{\Omega}} \varphi^{2\gamma_{k-1}} s^{2\beta_1^{k-1}} t^{2\beta_2^{k-1}} dsdt \\ &= \gamma_k^2 C_k,\end{aligned}$$

after noting

$$2(\gamma_k - 1) = 2\gamma_{k-1}, \quad 2\beta_1^{k-1} = 2\beta_1^k - (m-1), \quad 2\beta_2^{k-1} = 2\beta_2^k - (n-1).$$

Also note we can start the iteration since the first term is given by

$$\int_{\widehat{\Omega}} \varphi^{2^*} s^{m-1} t^{n-1} dsdt,$$

which is controlled by $\|\nabla\varphi\|_{L^2(\Omega)}$ by the classical critical Sobolev imbedding theorem. □

Corollary 5.2. *Let $m = n = \frac{N}{2}$ and suppose we have $1 < p \leq \frac{2N+2\alpha}{N-2}$ where $\alpha > 0$. Then we have $K_+ \subset L^p(B_1, |x|^\alpha dx)$ (ie. a continuous imbedding).*

Note the imbedding if optimal after considering the radial imbedding.

Proof of Corollary 5.2. We first prove the result for the case of $p = 2^* + 2k$ for some positive integer k and we suppose α satisfies the hypothesis. Let $\varphi \in K_+$ and we suppose $\|\nabla\varphi\|_{L^2(\Omega)} = 1$. By the symmetry of the function it is sufficient we bound the desired integral on $\{(s,t) \in \widehat{\Omega} : s > t\}$ which in polar coordinates corresponds to $\{(\theta, r) : 0 < \theta < \frac{\pi}{4}, 0 < r < 1\}$. Since $\varphi \in K_+$ we have

$$\varphi(r,\theta) \leq \frac{16}{\pi} \int_{\frac{3\pi}{16}}^{\frac{\pi}{4}} \varphi(r,\hat{\theta}) d\hat{\theta} \quad \text{for } 0 < \theta < \frac{\pi}{8},$$

and by Jensen's inequality we have (for $p = 2^* + 2k$)

$$\varphi(r,\theta)^p \leq \frac{16}{\pi} \int_{\frac{3\pi}{16}}^{\frac{\pi}{4}} \varphi(r,\hat{\theta})^p d\hat{\theta}.$$

Then note if we write out the $L^p(B_1, |x|^\alpha dx)$ norm of φ over the region corresponding to $0 < \theta < \frac{\pi}{8}$ we arrive at (note the extra power of r is from $dsdt = rdrd\theta$)

$$\int_0^1 \int_0^{\frac{\pi}{8}} \varphi(r,\theta)^p r^\alpha r^{2(n-1)} r \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta dr$$

but this is bounded above by

$$\frac{16}{\pi} \int_0^1 \int_{\frac{3\pi}{16}}^{\frac{\pi}{4}} \varphi(r,\hat{\theta})^p d\hat{\theta} r^{\alpha+2(n-1)+1} dr \int_0^{\frac{\pi}{8}} \cos^{n-1}(\theta) \sin^{n-1}(\theta) d\theta,$$

and hence we just need to control

$$\int_0^1 \int_{\frac{3\pi}{16}}^{\frac{\pi}{4}} \varphi(r,\theta)^p r^{\alpha+2(n-1)+1} d\theta dr. \tag{46}$$

We now show for all $0 < \theta_0 < \frac{\pi}{4}$ we can control

$$\int_0^1 \int_{\theta_0}^{\frac{\pi}{4}} \varphi(r,\theta)^p r^{\alpha+2(n-1)+1} d\theta dr. \tag{47}$$

We now write out (44) in terms of polar coordinates and noting the sine and cosine terms don't play a role now we see (44) gives the existence of some $D_k(\theta_0) > 0$ such that

$$\int_0^1 \int_{\theta_0}^{\frac{\pi}{4}} \varphi(r,\theta)^p r^{2(k+1)(n-1)+1} d\theta dr \leq D_k. \tag{48}$$

Note the assumption on α is exactly $2k(n-1) \leq \alpha$ and this gives us that

$$\alpha + 2(n-1) + 1 \geq 2(k+1)(n-1) + 1,$$

and hence we get the desired result for the case of $p = 2^* + 2k$.

We now prove the result for general p. Let p and α satisfy the hypothesis and we assume $p > 2^*$. First note that this assumption on p implies $p(n-1) - 2n > 0$. Since $p \leq \frac{2N+2\alpha}{N-2}$ we have $\alpha \geq p(n-1) - 2n$. Define $\alpha_p = p(n-1) - 2n$ and hence $p = \frac{2N+2\alpha_p}{N-2}$ so $p - 2^* = \frac{\alpha_p}{n-1}$. Pick k large integer such that $2^* + 2k > p$ and $2k(n-1) > \alpha_p$. Then set

$$\beta_k = \frac{(2^* + 2k)\alpha_p}{2k(n-1)}$$

and note $\beta_k < p$ for large k. Set $t_k = \frac{2^*+2k}{\beta_k} > 1$ for large k. Then note we have $t_k'(p - \beta_k) = 2^*$. Hence we have

$$\begin{aligned} \int_\Omega \varphi(x)^p |x|^{\alpha_p} dx &= \int_\Omega \varphi^{\beta_k} |x|^{\alpha_p} \varphi^{p-\beta_k} dx \\ &\leq \left(\int_\Omega \varphi^{\beta_k t_k} |x|^{\alpha_p t_k} dx \right)^{\frac{1}{t_k}} \left(\int_\Omega \varphi^{t_k'(p-\beta_k)} dx \right)^{\frac{1}{t_k'}} \\ &= \left(\int_\Omega \varphi^{2^*+2k} |x|^{2k(n-1)} dx \right)^{\frac{1}{t_k}} \left(\int_\Omega \varphi^{2^*} dx \right)^{\frac{1}{t_k'}} \end{aligned}$$

and this gives us the desired bound at least in the case of α_p. Noting that $\alpha \geq \alpha_p$ gives the desired result. □

Proposition 5.2. *(Pointwise invariance for the Hénon equation) Suppose N is even with $2m = 2n = N$, $u \in K_+$ and v solves*

$$\begin{cases} -\Delta v = |x|^\alpha u^{p-1} & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1. \end{cases} \tag{49}$$

Then $v \in K_+$.

Proof. The proof that proved the analagous result on an annulus works in this case also (the main difference is one needs to take some care near the origin now). In this proof we will write $\widehat{\Omega}, \widetilde{\Omega}, \widetilde{\Omega}_0$ even though its understood that $\Omega = B_1$. Let $u \in K_+$ and we perform the cut off as always $u_k(x) = \min\{u(x), k\}$ and we let v^k denote a solution of

$$\begin{cases} -\Delta v^k = |x|^\alpha u_k^{p-1} & \text{in } B_1, \\ v^k = 0 & \text{on } \partial B_1. \end{cases} \tag{50}$$

Writing this in term of polar coordinates gives

$$-v_{rr}^k - \frac{(N-1)v_r^k}{r} - \frac{v_{\theta\theta}^k}{r^2} + \frac{v_\theta^k}{r^2} h(\theta) = r^\alpha u_k^{p-1} = G(r,\theta), \quad \text{in } \widetilde{\Omega}, \tag{51}$$

where h is defined as in (40) with $m = n$ and note that G is even across $\theta = \frac{\pi}{4}$ after noting the conditions on u. Using the symmetry of v^k one sees, as in the case of the annulus, that $v_\theta^k = 0$ on $\theta = 0, \frac{\pi}{2}$ provided one stays away from the origin. As in the case of the annulus we consider $\widehat{v}(r,\theta) = v^k(r, \frac{\pi}{2} - \theta)$ and as before $\widehat{v}$ also satisfies (51) with the same boundary conditions as v^k. Set $\widehat{v}(x)$ to be $\widehat{v}(r,\theta)$ written in terms of x and we set $W(x) = v^k(x) - \widehat{v}(x)$. Then note $\Delta W(x) = 0$ in $B_1 \backslash \{0\}$ with $W = 0$ on ∂B_1 and since we are assuming the dimension $N \geq 3$ we can use the regularity of W to see that $W = 0$ and hence we have v^k is even across $\theta = \frac{\pi}{4}$ and hence we have $v_\theta^k(r, \frac{\pi}{4}) = 0$ for $0 < r \leq 1$.

Monotonicity. Let $w = v_\theta^k$ and then note that if we take a derivative in θ of the equation for v^k we arrive at

$$-w_{rr} - \frac{(N-1)w_r}{r} - \frac{w_{\theta\theta}}{r^2} + \frac{w_\theta}{r^2} h(\theta) + \frac{w}{r^2} h'(\theta) = \partial_\theta \left\{ r^\alpha u_k^{p-1} \right\}, \text{ in } \widetilde{\Omega}, \tag{52}$$

and in particular the equation is satisfied in $\widetilde{\Omega}_0$ with $w = 0$ on the portion of $\partial \widetilde{\Omega}_0$ corresponding to $\theta = 0, \frac{\pi}{4}$ and $w = 0$ on $r = 1$. As before a computation shows that if write the left hand side of (52) in terms of x we arrive at

$$-\Delta w(x) + \frac{(n-1)|x|^2 w(x)}{(x_1^2 + \cdots + x_m^2)(x_{m+1}^2 + \cdots + x_N^2)},$$

which, at least formally, satisfies a maximum principle. We can now proceed as in the annulus case to show that $w \geq 0$ in $\widetilde{\Omega}_0$; the only real difference is the added singularity at the origin. Note that w is Hölder continuous and there is some $C > 0$ such that $|w_\theta| \leq Cr$. This bound allows us to proceed as before using the method of [12] to see that $w \geq 0$ in $\widetilde{\Omega}_0$.

Sending $k \to \infty$. We can utilize the same arguments from the case of the annular domain in passing to the limit in k in Theorem 4.2. □

Proof of Theorem 5.1. Here again, we are going to use Theorem 2.1 for the proof. Note that conditions (i) and (ii) in Theorem 2.1 follows from Corollary 5.2 and Proposition 5.2 respectively. This proves the existence of a weak solution u of (43). It also follows from Theorem 8.1 that for

$$\frac{4(N+2)}{\beta_0(B_1)} < p - 2,$$

the ground state solution u obtained above is non-radial. Here $\beta_0(B_1)$ is the best constant for the Hardy inequality on B_1, and in fact $\beta_0(B_1) = (N-2)^2/4$. Thus, our solution u is non-radial provided

$$\frac{16(N+2)}{(N-2)^2} + 2 < p < \frac{2N + 2\alpha}{N-2}.$$

Regularity of the solution. Set $q := \frac{2N+2\alpha}{N-2}$ and consider $t_0 = 1$ and

$$t_{k+1} := \frac{q t_k}{2} - \frac{p-2}{2},$$

for $k \geq 0$. Since $1 < p < q$ we have, as before, $t_k \to \infty$. Let $u \in K_+$ denote the ground state and note then we have

$$\int_{B_1} |x|^\alpha u^{p+2(1)-2} dx = \int_{B_1} |\nabla u|^2 dx < \infty.$$

We now prove the following iteration:

$$\text{if} \int_{B_1} |x|^\alpha u^{p+2t_k-2} dx = C_k < \infty \quad \text{then} \int_{B_1} |x|^\alpha u^{p+2t_{k+1}-2} dx = D_k < \infty.$$

Fix $k \geq 0$ and suppose C_k is finite and then we consider

$$\varphi(x) = \begin{cases} u(x)^{2t_k-1} & \text{if } u(x) < i, \\ i^{2t_k-1} & \text{if } u(x) \geq i, \end{cases} \tag{53}$$

for positive integers i. This is a suitable test function to test the equation for u on and we then arrive at

$$(2t_k - 1) \int_{\Omega_i} u^{2t_k-2} |\nabla u|^2 dx = \int_{\Omega_i} |x|^\alpha u^{p+2t_k-2} dx + \varepsilon_{k,i},$$

where $\Omega_i := \{x \in B_1 : u(x) < i\}$ and

$$\varepsilon_{k,i} = \int_{\Omega_i^c} |x|^\alpha u^{p-1} i^{2t_k-1} dx,$$

where Ω_i^c is the compliment of Ω_i in B_1. We will later show that $\varepsilon_{k,i} \to \infty$ as $i \to \infty$ and hence lets accept this for now. Sending $i \to \infty$ in the above equality we arrive at

$$\frac{(2t_k - 1)}{t_k^2} \int_{B_1} |\nabla u^{t_k}|^2 dx = \int_{B_1} |x|^\alpha u^{p+2t_k-2} dx.$$

From this we see that $u^{t_k} \in H^1_{0,G}(B_1)$ and hence we see that $u^{t_k} \in K_+$. We can now use the continuous imbedding to see there is some $C = C_q$ such that

$$\frac{(2t_k - 1)C_q}{t_k^2} \left(\int_{B_1} |x|^\alpha u^{t_k q} dx \right)^{\frac{2}{q}} \leq \int_{B_1} |x|^\alpha u^{p+2t_k-2} dx = C_k,$$

but note that $qt_k = 2t_{k+1} + p - 2$ and hence we have $D_k < \infty$, which proves the inductive step. Since we have the result for t_0 we can start the iteration and hence we have C_k is finite for all k. Since $\alpha > 0$ we see that after a finite number of steps that $|x|^\alpha u^{p-1} \in L^T(B_1)$ for some $T > \frac{N}{2}$ and hence we have the solution is Hölder continuous. We can now use Schauder regularity theory to show the solution is a classical solution.

We now prove the claim that $\varepsilon_{k,i} \to 0$ as $i \to \infty$. First note that since

$$\int_{B_1} |x|^\alpha u^{p+2t_k-2} dx = C_k < \infty,$$

we have

$$\int_{\Omega_i^c} |x|^\alpha dx \leq \frac{C_k}{i^{p+2t_k-2}}. \tag{54}$$

Let $1 < \tau < \infty$ be such that $(p-1)\tau = p + 2t_k - 2$ and then note we have

$$\begin{aligned} \frac{\varepsilon_{k,i}}{i^{2t_k-1}} &= \int_{\Omega_i^c} |x|^{p-1} dx \\ &\leq \left(\int_{\Omega_i^c} |x|^\alpha u^{\tau(p-1)} dx \right)^{\frac{1}{\tau}} \left(\int_{\Omega_i^c} |x|^\alpha dx \right)^{\frac{1}{\tau'}} \end{aligned}$$

and put $\delta_i := \int_{\Omega_i^c} |x|^\alpha u^q dx$ and note $\delta_i \to 0$. So we can now use this and (54) to see that

$$\frac{\varepsilon_{k,i}^{\tau'}}{i^{\tau'(2t_k-1)}} \leq \delta_i^{\frac{\tau'}{\tau}} \frac{C_k}{i^{p+2t_k-2}},$$

and note the exponents on i are equal and hence we see that $\varepsilon_{k,i} \to 0$ as $i \to \infty$. □

6 Hénon equation with a zero order term on $\mathbb{R}^N$

In this section we examine solutions of

$$-\Delta u + u = |x|^\alpha u^{p-1} \qquad \text{in } \mathbb{R}^N = \mathbb{R}^n \times \mathbb{R}^n. \tag{55}$$

A particular interest will be in obtaining positive classical nonradial solutions. Before stating our main result we recall the definition of the best constant in Hardy inequality for $\mathbb{R}^N$, that is,

$$\beta_1 = \inf_{u \in H_0^1(\mathbb{R}^N)} \frac{\int_{\mathbb{R}^N} |\nabla u|^2 \, dx + \int_{\mathbb{R}^N} u^2 \, dx}{\int_{\mathbb{R}^N} \frac{u^2}{|x|^2} \, dx}. \tag{56}$$

Here is our main result in this section.

Theorem 6.1. *Let $N \geq 3$ be an even number and $\alpha > 0$. The following assertions hold:*

1. *Suppose*
$$\frac{2N + 2\alpha - 4}{N-2} < p < \frac{2N + 2\alpha}{N-2}.$$
Then there is a positive classical K_+ ground state solution u of (55) (see below for a definition of K_+).

2. *Suppose*
$$\max\left\{ \frac{4(N+2)}{\beta_1} + 2, \frac{2N + 2\alpha - 4}{N-2} \right\} < p < \frac{2N + 2\alpha}{N-2}.$$
Then the positive classical K_+ ground state solution u of (55) is nonradial.

Consider the full space $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ (here m and n need not be equal but later we will set them equal)

$$H_G^1(\mathbb{R}^N) := \left\{ u \in H^1(\mathbb{R}^N) : gu = u \quad \forall g \in G \right\},$$

where $gu(x)$ and $G := O(m) \times O(n)$ are as defined before. We now take $\widehat{\Omega}$ as before and hence in this case we have $\widehat{\Omega}$ is the first quadrant in the (s,t) plane. We define

$$\widetilde{\Omega} := \left\{ (\theta, r) : 0 < r < \infty, 0 < \theta < \frac{\pi}{2} \right\},$$

and we take

$$\widetilde{\Omega}_0 := \left\{ (\theta, r) : 0 < r < \infty, 0 < \theta < \frac{\pi}{4} \right\}.$$

We set K_+ where the definition has the added modifications to $\mathbb{R}^N$ that one would expect; so the functions are even across $\theta = \frac{\pi}{4}$ and increasing in θ on $(0, \frac{\pi}{4})$.

Proposition 6.1. *(Imbedding) For all integers $k \geq 0$ there is some C_k such that for all $0 \leq \varphi \in H^1_G(\mathbb{R}^N)$ with $\|\varphi\|_{H^1} \leq 1$ one has*

$$\int_{\widehat{\Omega}} (\varphi(s,t))^{2^*+2k} s^{(k+1)(m-1)} t^{(k+1)(n-1)} dsdt \leq C_k, \tag{57}$$

$$\int_{\widehat{\Omega}} (\varphi(s,t))^{2(k+1)} s^{(k+1)(m-1)} t^{(k+1)(n-1)} dsdt \leq C_k. \tag{58}$$

Proof. Both results will start with the same basic proof and they will follow by almost the same computation as the proof of Proposition 5.1. By a density argument we can assume $\varphi \geq 0$ is smooth and zero for large enough $r = (s^2+t^2)^{\frac{1}{2}}$ and we write $d\mu(s,t) = s^{m-1}t^{n-1}dsdt$. Suppose

$$\int_{\widehat{\Omega}} \left(|\nabla_{s,t}\varphi|^2 + \varphi^2 \right) d\mu(s,t) \leq 1.$$

Let $0 \leq u$ denote a function defined in (s,t) and zero for large (s,t) (we will take u to be a power of φ). As before we have

$$u(s,t)^2 \leq \int_s^\infty |\nabla_{s,t}u(\tau_1,t)| d\tau_1 \int_t^\infty |\nabla_{s,t}u(s,\tau_2)| d\tau_2,$$

and we now multiply by sides by $s^{2\beta_1}t^{2\beta_2}$ where $\beta_i > 0$ and integrate over $\widehat{\Omega}$ we arrive at

$$\int_{\widehat{\Omega}} u(s,t)^2 s^{2\beta_1} t^{2\beta_2} dsdt \leq \left(\int_{\widehat{\Omega}} |\nabla_{s,t}u(s,t)| s^{\beta_1} t^{\beta_2} dsdt \right)^2. \tag{59}$$

We now suppose $0 \leq \varphi$ as above and take $u = \varphi^\gamma$ and put into abouve $(\gamma \geq 1)$. Then we arrive at

$$\begin{aligned} \int_{\widehat{\Omega}} \varphi^{2\gamma} s^{2\beta_1} t^{2\beta_2} dsdt &\leq \gamma^2 \left(\int_{\widehat{\Omega}} \left\{ |\nabla_{s,t}\varphi| s^{\frac{m-1}{2}} t^{\frac{n-1}{2}} \right\} \left\{ \varphi^{\gamma-1} s^{\beta_1 - \frac{m-1}{2}} t^{\beta_2 - \frac{n-1}{2}} \right\} dsdt \right)^2 \\ &\leq \gamma^2 \|\nabla\varphi\|^2_{L^2(\Omega)} \int_{\widehat{\Omega}} \varphi^{2(\gamma-1)} s^{2\beta_1-(m-1)} t^{2\beta_2-(n-1)} dsdt, \end{aligned}$$

where we performed the Cauchy–Schwarz inequality and recall $\|\nabla\varphi\|_{L^2} \leq 1$.

We now perform the iterations. For (57) we will follow the exact same choice of parameters as in Proposition 5.1 and this gives the desired result. Note in the first step here we choose the parameters so that the right hand side is exactly

$$\int_{\widehat{\Omega}} \varphi^{2^*} s^{m-1} t^{n-1} dsds,$$

which we know is controlled by the critical Sobolev imbedding.

To prove (58) the only difference is we choose the parameters so that in the first step of the iteration the right hand side is

$$\int_{\widehat{\Omega}} \varphi^2 s^{m-1} t^{n-1} ds dt.$$

If one performs the iteration they get the desired result.

□

Corollary 6.1. *For $N \geq 3$ even, $\alpha > 0$ and*

$$\frac{2N+2\alpha-4}{N-2} < p < \frac{2N+2\alpha}{N-2}, \tag{60}$$

we have $K_+ \subset\subset L^p(\mathbb{R}^N, |x|^\alpha dx)$.

Proof. Let $\varphi \in K_+$. Using a suitable compactly supported radial cut off function and Corollary 5.2 we see that there is some C (independent of φ) such that

$$\left(\int_{B_1} \varphi(x)^p |x|^\alpha dx \right)^{\frac{1}{p}} \leq C \|\varphi\|_{H^1(\mathbb{R}^N)},$$

and hence we really only need to bound the integral on the region $|x| \geq 1$.

By using Proposition 6.1 and similar arguments that we used to prove Corollary 5.2 we can show for all integers $k, i \geq 0$ there is some constant depending just on k, i such that for all $\varphi \in K_+$ with $\|\varphi\|_{H^1} \leq 1$ one has

$$\int_{\mathbb{R}^N} \varphi^{2+2i} |x|^{2i(n-1)} dx \leq C_i, \tag{61}$$

$$\int_{\mathbb{R}^N} \varphi^{2^*+2k} |x|^{2k(n-1)} dx \leq C_k. \tag{62}$$

Recall we really only need the estimate on the region $\frac{\pi}{8} < \theta < \frac{\pi}{4}$ (where s and t are comparable) and then we can extend to the full region via monotonicity and symmetry. We now interpolate between these to get the desired result. Again we fix $\varphi \in K_+$ with $\|\varphi\|_{H^1} \leq 1$. Then we have, for $\tau > 1$,

$$\begin{aligned} \int_{|x|>1} \varphi^p |x|^\alpha dx &= \int_{|x|>1} \left\{ \varphi^{\frac{(2+2i)}{\tau}} |x|^{\frac{2i(n-1)}{\tau}} \right\} \left(\varphi^{p-\frac{(2+2i)}{\tau}} |x|^{\alpha - \frac{2i(n-1)}{\tau}} \right) dx \\ &\leq \left(\int_{|x|>1} \varphi^{2+2i} |x|^{2i(n-1)} dx \right)^{\frac{1}{\tau}} \left(\int_{|x|>1} \varphi^{\tau'\left(p - \frac{2+2i}{\tau}\right)} |x|^{\tau'\left(\alpha - \frac{2i(n-1)}{\tau}\right)} dx \right)^{\frac{1}{\tau'}}. \end{aligned}$$

We now choose an appropriate τ and we will be more general than we need to. Assume $i, k \geq 0$ are integers and we suppose $2 + 2i < p < 2^* + 2k$. Take $\tau > 1$ such that

$$\tau' \left(p - \frac{2+2i}{\tau} \right) = 2^* + 2k,$$

and then note we have an estimate provided

$$\tau' \left(\alpha - \frac{2i(n-1)}{\tau} \right) \leq 2k(n-1),$$

after considering (62). Note on can explicitly compute τ from the first equation to get

$$\tau = \frac{2^* + 2k - 2 - 2i}{2^* + 2k - p}.$$

Now one needs to check if the second inequality holds. For our purposes it will be sufficient to take $i = 0$ and k large. So define τ_k by

$$\tau_k = \frac{2^* + 2k - 2}{2^* + 2k - p},$$

and so note that $\tau_k \searrow 1$ as $k \to \infty$. So we need $2 < p < 2^* + 2k$ and $\tau_k' \alpha \le 2k(n-1)$ which we can rewrite as $\frac{\alpha \tau_k}{2(n-1)} \le k(\tau_k - 1)$ but note that

$$k(\tau_k - 1) = \frac{k(p-2)}{2^* + 2k - p} \to \frac{p-2}{2},$$

as $k \to \infty$. So we see the desired result holds for large integers k provided

$$\frac{\alpha}{2(n-1)} < \frac{p-2}{2},$$

which is exactly the lower bound on p from (60). The above shows that for the desired range of parameters we have a continuous imbedding. We now need to improve this to a compact imbedding. Note the only potential loss of compactness is if we lose mass at ∞. Take $\{\varphi_k\}_k \subset K_+$ with $\|\varphi_k\|_{H^1} \le 1$ and fix p, α as in (60) and then note by taking $\varepsilon > 0$ small enough we have $p, \alpha_\varepsilon := \alpha + \varepsilon$ still satisfies (60). Then from the above results we have for some $\varepsilon > 0$ that

$$\int_{|x|>1} \varphi_k^p |x|^{\alpha+\varepsilon} dx \le C_\varepsilon,$$

and hence for large R we have

$$\int_{|x|>R} \varphi_k^p |x|^{\alpha} dx \le \frac{C_\varepsilon}{R^\varepsilon},$$

for all k and this is sufficient to rule out a loss of compactness at ∞. □

We now turn to the pointwise invariance property. We need to show that given $u \in K_+$ there is some $v \in K_+$ which satisfies

$$-\Delta v + v = |x|^\alpha u^{p-1} \text{ in } \mathbb{R}^N, \tag{63}$$

Proposition 6.2. *(Pointwise invariance property) Suppose* $2m = 2n = N$ *and* $u \in K_+$. *Then there is some* $v \in K_+$ *which satisfies (63).*

Proof. Let $u \in K_+$ and for integers $i \ge 1$ consider the problem

$$\begin{cases} -\Delta v + v = |x|^\alpha u^{p-1} & \text{in } B_i, \\ v = 0 & \text{on } \partial B_i. \end{cases} \tag{64}$$

Using the same proof as in Proposition 5.2 we can show there is some $v_i \in K_+(B_i)$, where $K_+(B_i)$ is the obvious extension of K_+ from the unit ball to the ball of radius i, which satisfies (64). Extend

v_i to be zero outside B_i. Then note by multiplying the equation for v_i and integrating over B_i we obtain

$$\begin{aligned}\int_{B_i} |\nabla v_i|^2 + v_i^2 dx &= \int_{B_i} |x|^\alpha u^{p-1} v_i dx \\ &= \int_{B_i} \left\{ u^{p-1} |x|^{\frac{\alpha}{p'}} \right\} \left\{ v_i |x|^{\frac{\alpha}{p}} \right\} dx \\ &\le \left(\int_{B_i} u^{p'(p-1)} |x|^\alpha dx \right)^{\frac{1}{p'}} \left(\int_{B_i} v_i^p |x|^\alpha dx \right)^{\frac{1}{p}}\end{aligned}$$

and then note $p'(p-1) = p$. Using the imbedding from Corollary 6.1 there is some C such that we have

$$\int_{\mathbb{R}^N} |\nabla v_i|^2 + v_i^2 dx \le C \|u\|_{H^1(\mathbb{R}^N)}^{p-1} \|v_i\|_{H^1(\mathbb{R}^N)},$$

and this shows that $\{v_i\}_i$ is bounded in $H^1(\mathbb{R}^N)$. By passing to a sequence we can assume there is some $v \in H^1(\mathbb{R}^N)$ with $v_i \rightharpoonup v$ in $H^1(\mathbb{R}^N)$ and v is an $H^1(\mathbb{R}^N)$ energy solution of (63). Furthermore we can use arguments similar to before to show that $v \in K_+$, we omit the details.

□

Proof of Theorem 6.1. Here again, we are going to use Theorem 2.1 for the proof. Note that conditions (i) and (ii) in Theorem 2.1 follows from Corollary 6.1 and Proposition 6.2 respectively. This proves the existence of a weak solution u of (55). It also follows from Theorem 8.1 that for

$$\frac{4(N+2)}{\beta_1} < p - 2,$$

the ground state solution u obtained above is non-radial. Here β_1 is the best constant for the Hardy inequality on $\mathbb{R}^N$ defined in (56). Thus, our solution u is non-radial provided

$$\max\left\{ \frac{4(N+2)}{\beta_1} + 2, \frac{2N + 2\alpha - 4}{N-2} \right\} < p < \frac{2N+2\alpha}{N-2}.$$

Regularity of the solution. Here we can use a proof similar to the proof of Theorem 5.1 but one needs to insert a suitable cut off function. We omit the details. □

7 A singular potential problem

Here we examine the problem

$$\begin{cases} -\Delta u + \frac{u}{|x|^\alpha} = u^{p-1} & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{65}$$

where $N \ge 3$ and $\alpha > 2$. In particular we are interested in nonradial positive classical solutions. Note that we are taking $\alpha > 2$ which can be thought of as super critical values of α. Let H denote the completion of the $\{u \in C_c^\infty(B_1 \backslash \{0\}) : u = u(s,t)\}$ under the norm

$$\|u\|_H^2 := \int_\Omega |\nabla u|^2 + \frac{u^2}{|x|^\alpha} dx.$$

Note if $\alpha \ge N$ then H does not contain $C_c^\infty(B_1)$ and hence we need to be a bit careful when we define what we mean by a solution.

Definition 7.1. *We call u a weak H solution of*

$$\begin{cases} -\Delta u + \frac{u}{|x|^\alpha} = f(x) & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{66}$$

provided $u \in H$ and

$$\int_{B_1} \left(\nabla u \cdot \nabla \varphi + \frac{u\varphi}{|x|^\alpha} \right) dx = \int_{B_1} f(x)\varphi dx \quad \forall \varphi \in H. \tag{67}$$

We will assume that we are in the case of $m = n$ since we will want to work on a suitable version of K_+ which we now define. We define K_+ to be exactly analogous to the way it was defined for the Hénon problem on the ball except now we add the extra condition that $u \in H$.

Theorem 7.1. *Suppose $m = n$ and consider the problem (65). The following assertions hold;*

1. *Suppose $2 < p < \frac{2N+2\alpha-4}{N-2}$, then there is a positive classical K_+ ground state solution of (65). In addition for all $t > 0$ there is some C_t such that $u(x) \le C_t |x|^t$ in B_1.*
2. *The ground state solution from part 1 is nonradial provided*

$$p - 2 > 4(N+2)/\beta_\alpha(\Omega),$$

where

$$\beta_\alpha := \inf_{0 \neq \varphi \in H} \frac{\int_{B_1} |\nabla \varphi|^2 + \frac{\varphi^2}{|x|^\alpha} dx}{\int_{B_1} \frac{\varphi^2}{|x|^2} dx}.$$

We will show $\beta_\alpha \to \infty$ as $\alpha \to \infty$ and hence this result is nonempty.

Remark 7.2. *We are able to prove similar results for nonradial domains provided they are domains of double revolution symmetry with the $\pi/2$ or $\pi/4$ symmetry and the needed monotonicity. In these cases one works on a suitable version of K_- but we chose not to include these results since the imbeddings we are able to prove appear to be nonoptimal.*

Lemma 7.3. *We have $\lim_{\alpha \to \infty} \beta_\alpha = \infty$.*

Proof. Recall the boundary Hardy inequality gives

$$\int_{B_1} |\nabla \varphi|^2 dx \ge \frac{1}{4} \int_{B_1} \frac{\varphi^2}{(1-|x|)^2} dx \quad \forall \varphi \in H_0^1(B_1).$$

Define

$$H_\alpha(r) := r^2 \left(\frac{1}{4(1-r)^2} + \frac{1}{r^\alpha} \right) \quad 0 < r < 1,$$

and we set $C_\alpha = \min_{0<r<1} H_\alpha(r)$ and note $C_\alpha \to \infty$ as $\alpha \to \infty$. Then note we have

$$\begin{aligned} \int_{B_1} |\nabla \varphi|^2 dx + \int_{B_1} \frac{\varphi^2}{|x|^\alpha} dx - C_\alpha \int_{B_1} \frac{\varphi^2}{|x|^2} dx &\ge \int_{B_1} \left(\frac{1}{4(1-|x|)^2} + \frac{1}{|x|^\alpha} - \frac{C_\alpha}{|x|^2} \right) \varphi^2 dx \\ &= \int_{B_1} \frac{\varphi^2}{|x|^2} \left(\frac{|x|^2}{4(1-|x|)^2} + \frac{|x|^2}{|x|^\alpha} - C_\alpha \right) dx \\ &= \int_{B_1} \frac{\varphi^2}{|x|^2} \left(H_\alpha(|x|) - C_\alpha \right) dx \\ &\ge 0, \end{aligned}$$

and from this we see that $\beta_\alpha \ge C_\alpha$ which proves the desired result.

□

Lemma 7.4. *Suppose $m = n$ and $1 \le p < \frac{2N+2\alpha-4}{N-2}$. Then $K_+ \subset\subset L^p(B_1)$.*

Proof. Suppose in addition to the hypothesis on p take $p > 2$ and then for $u \in K_+$ with $\|u\|_H = 1$ and $1 < \tau < \infty$ we have

$$\begin{aligned} \int_{B_1} u^p dx &= \int_{B_1} \frac{u^{\frac{2}{\tau}}}{|x|^{\frac{\alpha}{\tau}}} \left\{ u^{p-\frac{2}{\tau}} |x|^{\frac{\alpha}{\tau}} \right\} dx \\ &\le \|u\|_H^{\frac{2}{\tau}} \left(\int_{B_1} u^{\tau'(p-\frac{2}{\tau})} |x|^{\frac{\tau'\alpha}{\tau}} dx \right)^{\frac{1}{\tau'}} \end{aligned}$$

and now note we need to have some sort of Hénon type imbedding for K_+. Note this same proof so far would work on a general domain with a function with any type of symmetry. By Corollary 5.2 we see this integral on the right is bounded by a constant provided we have

$$\tau'\left(p - \frac{2}{\tau}\right) < \frac{2N + 2\frac{\tau'\alpha}{\tau}}{N-2}. \tag{68}$$

Since $p < \frac{2N+2\alpha-4}{N-2}$ we can show (68) for τ sufficiently close to 1 and this completes the proof of the continuity of the imbedding. For compactness we use compactness in $L^1(B_1)$ along with standard L^p interpolation. □

In the proof of the above Lemma it is apparent that once one has a type of Hénon imbedding then they get an suitable imbedding for H.

Proposition 7.1. *(Pointwise Invariance) Take $m = n$ and suppose $u \in K_+$. Then there is some $v \in K_+$ which solves*

$$\begin{cases} -\Delta v + \frac{v}{|x|^\alpha} = u^{p-1} & \text{in } B_1, \\ v = 0 & \text{on } \partial B_1. \end{cases} \tag{69}$$

Proof. Our approach will be to approximate the domain via an annulus and take a limit. For $\varepsilon > 0$ small set $A_\varepsilon := \{x \in \mathbb{R}^N : \varepsilon < |x| < 1\}$ and let $u \in K_+$. Consider the problem

$$\begin{cases} -\Delta v^\varepsilon + \frac{v^\varepsilon}{|x|^\alpha} = u^{p-1} & \text{in } A_\varepsilon, \\ v^\varepsilon = 0 & \text{on } \partial A_\varepsilon. \end{cases} \tag{70}$$

Note this problem essentially fits into the exact framework of Proposition 4.2 part 2 except for this $|x|^\alpha$ term; but this term has no effect on the approach. So if we let $K_+(A_\varepsilon)$ denote the obvious extension of K_+ to A_ε we see that $v^\varepsilon \in K_+(A_\varepsilon)$. We now extend v^ε to B_1 by extending it to be zero outside A_ε and note $v_\varepsilon \in K_+$. Then note we

$$\begin{aligned} \int_{B_1} |\nabla v^\varepsilon|^2 + \int_{B_1} \frac{(v^\varepsilon)^2}{|x|^\alpha} dx &= \int_{B_1} u^{p-1} v^\varepsilon dx \\ &\le \left(\int_{B_1} u^{p'(p-1)} dx \right)^{\frac{1}{p'}} \left(\int_{B_1} (v^\varepsilon)^p dx \right)^{\frac{1}{p}} \\ &\le \|u\|_{L^p(B_1)}^{\frac{p}{p'}} C \|v^\varepsilon\|_H \end{aligned}$$

where in the last step we used the imbedding of K_+. From this we see there is some constant C_u such that $\|v^\varepsilon\|_H \le C_u$ for all $\varepsilon > 0$ small. From this we see we can pass to a subsequence and find some $v \in H$ such that $v^\varepsilon \rightharpoonup v$ in H and also since K_+ is convex and closed in H we have it weakly closed in H and hence $v \in K_+$. Note if $\varphi \in C_c^\infty(B_1 \backslash \{0\})$ we can easily pass to the limit in

$$\int_{B_1} \nabla v^\varepsilon \cdot \nabla \varphi + \frac{v^\varepsilon \varphi}{|x|^\alpha} dx = \int_{B_1} u^{p-1} \varphi dx, \tag{71}$$

and hence we have a solution at least on the punctured ball. □

Proof of Theorem 7.1. We shall begin by observing that Theorem 2.1 can be easily adapted to deal with singular problems like (65). The only major change is to replace the notion of the weak solutions in condition (ii) of Theorem 2.1 by the one in Definition 7.1 where the test functions φ belong to the space $H \cap L^p(B_1)$. Both conditions (i) and (ii) follow from Lemma 7.4 and Proposition 7.1 respectively. This proves the existence of a weak solution for (65). Also, a similar argument as in the proof of Theorem 8.1 shows that the solution is nonradial provided

$$p - 2 > 4(N+2)/\beta_\alpha(\Omega).$$

Moreover, by Lemma 7.3 we have that $\beta_\alpha \to \infty$ as $\alpha \to \infty$ and hence the ground state solution u is non-radial for large values of α.

Regularity of ground state solution. Let $u \in K_+$ denote a ground state solution of (65) and note since $2 < p < \frac{2N+2\alpha-4}{N-2}$ there is some $t_0 > 1$ such that $u \in L^{p+2t_0-2}(B_1)$ (after considering the imbedding result). For $k \geq 0$ define

$$t_{k+1} = \frac{pt_k}{2} - \frac{(p-2)}{2},$$

and note that $t_k \nearrow \infty$ as $k \to \infty$. We will now show one has the following iteration result: for $k \geq 0$

$$\text{if } u \in L^{p+2t_k-2}(B_1) \quad \text{then} \quad u \in L^{p+2t_{k+1}-2}(B_1). \tag{72}$$

We now prove this iteration step; let $k \geq 0$ and suppose $u \in L^{p+2t_k-2}(B_1)$. For $m \geq 1$ set

$$\varphi_m(x) = \begin{cases} u(x)^{2t_k-1} & \text{if } u(x) < m, \\ u(x)m^{2t_k-2} & \text{if } u(x) \geq m, \end{cases} \tag{73}$$

and since $2t_k - 1 > 1$ we see that $\varphi_m \in H_0^1(B_1)$ and its also clear that we have $\int_{B_1} |x|^{-\alpha} \varphi_m^2 dx < \infty$ and hence $\varphi_m \in H$, so we can use φ_m as a test function in the definition of u be a weak H solution of (65) to arrive at (after dropping a couple of positive terms from the left)

$$\begin{aligned} \frac{(2t_k-1)}{t_k^2} \int_{\Omega_m} |\nabla u^{t_k}|^2 dx + \int_{\Omega_m} \frac{u^{2t_k}}{|x|^\alpha} dx \quad &\leq \quad \int_{\Omega_m} u^{p+2t_k-2} dx \\ &\quad + m^{2t_k-2} \int_{\Omega_m^c} u^p dx \end{aligned}$$

where $\Omega_m = \{x \in B_1 : u(x) < m\}$ and Ω_m^c is its compliment in B_1. Set $\varepsilon_m := m^{2t_k-2} \int_{\Omega_m^c} u^p dx$ and we will later show that $\varepsilon_m \to 0$ as $m \to \infty$. Then note passing to the limit in the above inequality we arrive at

$$\frac{(2t_k-1)}{t_k^2} \int_{B_1} |\nabla u^{t_k}|^2 dx + \int_{B_1} \frac{u^{2t_k}}{|x|^\alpha} dx \leq \int_{B_1} u^{p+2t_k-2} dx, \tag{74}$$

and note the integral on the right is finite since we have $u \in L^{p+2t_k-2}(B_1)$ by hypothesis. From this we see that $u^{t_k} \in H$ and note that u^{t_k} and now its easy to see that $u^{t_k} \in K_+$ and hence by the imbedding result we have $u^{t_k} \in L^p(B_1)$ but note $t_k p = 2t_{k+1} + p - 2$ and hence we have proven the iteration step. We now show $\varepsilon_m \to 0$. By hypothesis we have $u \in L^{p+2t_k-2}(B_1)$ and hence $\delta_m := \int_{\Omega_m^c} u^{p+2t_k-2} dx \to 0$ and note that $\varepsilon_m \le \delta_m$ which gives the desired result.

With this iteration we have $u \in L^T(B_1)$ for all $1 < T < \infty$. At this point we could attempt to appeal to some linear theory to show u is bounded but we prefer to follow the iteration through. Once we have u bounded then we will switch to linear theory.

Starting at (74) and dropping a portion of the zero order part of the norm we arrive

$$\|u^{t_k}\|_H^2 \le \frac{t_k^2}{2t_k - 1} \int_{B_1} u^{p+2t_k-2} dx,$$

and using the imbedding of K_+ into $L^p(B_1)$ we arrive at

$$\|u\|_{L^{p+2t_{k+1}-2}} \le \left(\frac{C_0 t_k^2}{2t_k - 1} \right)^{\frac{1}{2t_k}} \|u\|_{L^{p+2t_k-2}}^{\frac{p+2t_k-2}{2t_k}},$$

where C_0 is coming from the imbedding. We write

$$\beta_k := \|u\|_{L^{p+2t_k-2}}, \quad \gamma_k := \left(\frac{C_0 t_k^2}{2t_k - 1} \right)^{\frac{1}{2t_k}}, \quad \delta_k := \frac{p + 2t_k - 2}{2t_k}$$

and hence we have

$$\beta_{k+1} \le \gamma_k \beta_k^{\delta_k},$$

for all $k \ge 0$. Writing out the iteration we arrive at

$$\beta_{n+1} \le \left(\beta_0^{\prod_{j=0}^n \delta_j} \right) \prod_{k=0}^n \left(\gamma_k^{\prod_{i=k}^n \delta_i} \right).$$

We now wish to show the right hand side in bounded in n and hence this would give us the desired L^∞ bound on u. We first show that $T_n := \prod_{j=0}^n \delta_j$ is bounded. Consider the log of T_n and note we have

$$\begin{aligned} \ln(T_n) &= \sum_{j=0}^n \ln(\delta_j) \\ &= \sum_{j=0}^n \ln\left(1 + \frac{p-2}{2t_j}\right) \\ &\le \sum_{j=0}^n \frac{p-2}{2t_j} \end{aligned}$$

where we used the fact that $p > 2$ and log is concave. Now note one can get the explicit formula $t_k = C\left(\frac{p}{2}\right)^k + 1$ where $C > 0$ since $t_0 > 1$. From this we see that $\ln(T_n)$ is bounded and hence we have the same for T_n. We now define $T_{k,n} := \prod_{i=k}^n \delta_i$ and similarly we get

$$\begin{aligned} \ln(T_{k,n}) &= \sum_{i=k}^n \ln\left(1 + \frac{p-2}{2 + 2C2^{-i}p^i}\right) \\ &\le \sum_{i=k}^n \frac{p-2}{2 + 2C2^{-i}p^i} \\ &\le \frac{\hat{C}2^k}{p^k} \end{aligned}$$

for some $\hat{C}$ independent of k and n. This shows $T_{k,n}$ is bounded above. From this we see that to show $\prod_{k=0}^{n}\left(\gamma_k^{\prod_{i=k}^{n}\delta_i}\right)$ is bounded it is sufficient to show that $P_n := \sum_{k=0}^{n}\ln(\gamma_k)$ is bounded. But note that

$$P_n = \sum_{k=0}^{n}\frac{1}{2t_k}\ln\left(\frac{C_0 t_k^2}{2t_k-1}\right),$$

and noting the growth of t_k we easily see this is bounded in n. This completes the proof that u is bounded.

We will now apply Proposition 7.2 to get more regularity. Take $t<0$ but very close and then note that $u^{p-1}\in Y_t$ and by uniqueness of the solution to the linear problem we have $u\in X_t$ and hence we have $|u(x)|\le C|x|^{t+\alpha}$. We can now iterate this process. For instance we have $|u(x)^{p-1}|\le C|x|^{(p-1)(t+\alpha)}|$ and we choose $t_1:=(p-1)(t+\alpha)$ and apply the linear theory again to see that $|u(x)|\le C|x|^{(p-1)(t+\alpha)+\alpha}$. Writing out the iteration we see that for all $t>0$ there is some $C_t>0$ such that $u(x)\le C_t|x|^t$.

□

We now state a result from the preprint [2] but we include a partial proof for the readers convenience. This result will only be used when showing the decay of the solution near the origin.

Proposition 7.2. *[2] (Linear theory for $-\Delta\varphi+\frac{\varphi}{|x|^\alpha}$ in weighted L^∞ spaces) For $N\ge 3,\alpha>2$ and $t\in\mathbb{R}$ define the norms*

$$\|f\|_{Y_t} := \sup_{0<|x|\le 1}|x|^{-t}|f(x)|, \qquad \|\varphi\|_{X_t} := \sup_{0<|x|\le 1}|x|^{-t-\alpha}|\varphi(x)|,$$

we let Y_t denote the completion of the bounded functions under the Y_t norm and X_t to denote the continuous functions on $\overline{B_1}\backslash\{0\}$ which have finite X_t norm and with $\varphi=0$ on ∂B_1. Let $N\ge 3,\alpha>2$ and $t\in\mathbb{R}$. Then there is some $C>0$ such that for all $f\in Y_t$ there is a $\varphi\in X_t$ such that

$$\begin{cases} -\Delta\varphi(x)+\frac{\varphi(x)}{|x|^\alpha} = f(x) & \text{in } B_1\backslash\{0\},\\ \varphi = 0 & \text{on } \partial B_1,\end{cases} \tag{75}$$

and one has the estimate $\|\varphi\|_{X_t}\le C\|f\|_{Y_t}$. For

$$t > -\alpha - \frac{\left\{N-2+\sqrt{N^2-4N+8}\right\}}{2}, \tag{76}$$

the solution φ is unique.

Proof. Fix N,α and t as in the hypothesis. Let $f\in Y_t$ with $\|f\|_{Y_t}=1$. Since $\alpha>2$ we can fix $0<\varepsilon'<\frac{1}{4}$ small such that

$$1-(t+\alpha)(t+\alpha-1)|x|^{\alpha-2}-(N-1)(t+\alpha)|x|^{\alpha-2}\ge\frac{1}{2} \qquad \forall\, 0<|x|\le\varepsilon',$$

and note ε' only depends on N,α and t. We can now choose $C_i=C_i(N,\alpha,t)>0$ such that

$$C_1\left\{1-(t+\alpha)(t+\alpha-1)|x|^{\alpha-2}-(N-1)(t+\alpha)|x|^{\alpha-2}\right\}+C_2\frac{\left\{2N+\frac{1-|x|^2}{|x|^\alpha}\right\}}{|x|^t}\ge 1,\ \ \forall\, 0<|x|<1.$$

For $R_1<R_2$ we set $A_{R_1,R_2}:=\{x\in\mathbb{R}^N : R_1<|x|<R_2\}$. For $0<\varepsilon<\frac{\varepsilon'}{2}$ consider

$$\begin{cases} -\Delta\varphi_\varepsilon(x)+\frac{\varphi_\varepsilon(x)}{|x|^\alpha} = f(x) & \text{in } A_{\varepsilon,1},\\ \varphi_\varepsilon = 0 & \text{on } \partial A_{\varepsilon,1},\end{cases} \tag{77}$$

and note there is a classical solution. Set $\overline{\varphi}(x) := C_1|x|^{t+\beta} + C_2(1-|x|^2)$ and by the maximum principle we have $|\varphi_\varepsilon(x)| \le \overline{\varphi}$ in $A_{\varepsilon,1}$ for all small ε (note ε' is fixed and we be varying ε). In particular there is some $C_3 > 0$ such that $\sup_{A_{\frac{\varepsilon'}{2},1}} |\varphi_\varepsilon| \le C_3$ for all small $\varepsilon > 0$. We now set $\overline{\psi}(x) := C_4|x|^{t+\alpha}$ where $C_4 = C_3 + 2$. Then we can apply the maximum principle on $A_{\varepsilon,\varepsilon'}$ to see that $|\varphi_\varepsilon(x)| \le \psi(x) = C_4|x|^{t+\alpha}$ in $A_{\varepsilon,\varepsilon'}$. This shows that there is some $C > 0$ such that for all small $\varepsilon > 0$ we have $\|\varphi_\varepsilon\|_{X_t} \le C\|f\|_{Y_t}$ (where the norms are now over the annulus). The main point is the constant C does not depend on ε. Taking $\varepsilon = \varepsilon_m \searrow 0$ and applying a diagonal argument (using the equation to obtain the needed compactness away from the origin) there is some $\varphi \in X_t$ which solves (75) and we have the desired estimate.

We now prove the uniqueness part. Let $\varphi \in X_t$ solve (75) with $f = 0$. We write $\varphi(x) = \sum_{k=0}^\infty a_k(r)\psi_k(\theta)$ where (ψ_k, λ_k) are the eigenpairs of the Laplace-Beltrami operator $-\Delta_\theta = -\Delta_{S^{N-1}}$ on the unit sphere S^{N-1}. Then for all $k \ge 0$ we have a_k satisfies

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + \frac{\lambda_k a_k}{r^2} + \frac{a_k(r)}{r^\alpha} = 0 \qquad 0 < r < 1, \tag{78}$$

with $a_k(1) = 0$ and $|a_k(r)| \le C_k r^{t+\alpha}$. We now need to show that $a_k = 0$ for all $k \ge 0$. Take $w(r) := r^\gamma a_k(r)$ where $t + \alpha + \gamma > 0$. Then note we have $w(1) = 0 = \lim_{r\searrow 0} w(r)$ and hence if w is not identically zero we can (after multiplying by -1) see that w attains its max at some $0 < r_0 < 1$ with $w(r_0) > 0$, $w''(r_0) \le 0$ and $w'(r_0) = 0$. Note the equation for w is given by

$$w''(r) + \left(\frac{N-1}{r} - \frac{2\gamma}{r}\right) w'(r) + C_k(r)w(r), \quad 0 < r < 1,$$

where

$$C_k(r) = \frac{\gamma(\gamma+1)}{r^2} - \frac{\gamma(N-1)}{r^2} - \frac{\lambda_k^2}{r^2} - \frac{1}{r^\alpha}.$$

Note if $C_k(r_0) < 0$ then evaluating the equation for w at r_0 gives a contradiction. Now note that

$$r^2 C_k(r_0) < \gamma(\gamma+1) - \gamma(N-1) - 1,$$

and hence we have the desired contradiction provided $\gamma(\gamma+1) - \gamma(N-1) - 1 \le 0$. Let $\gamma_- < \gamma_+$ denote the roots of this quadratic equation and note we need some γ such that $t + \alpha + \gamma > 0$ and $\gamma \in (\gamma_-, \gamma_+)$. So to find such a γ it is sufficient that $t + \alpha + \gamma_+ > 0$ and writing this out gives (76). □

8 Nonradial solutions when Ω is a radial domain.

In this section we discuss the case when $a(x) = a(|x|)$ is radial, and Ω is a radial domain, that is $\Omega = \{x : R_1 \le |x| < R_2\}$ where $R_1 \ge 0$ and $R_2 \in (R_1, +\infty]$,

$$\begin{cases} -\Delta u + \lambda u = a(|x|)u^{p-1} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{79}$$

where $\lambda = 0$ for bounded domains and $\lambda = 1$ where $\Omega = \mathbb{R}^N$. Note we are writing a general form that can handle all radial domains we consider. When $R_1 = 0$ then we are either on a ball or the full space. When $R_1 > 0$ then we are taking R_2 finite (we are not examining exterior domains here) and then we should take $\Omega := \{x : R_1 < |x| < R_2\}$. We shall prove that the solution obtained in Theorem 4.1, Theorem 5.1 and Theorem 6.1 are nonradial under certain assumptions on Ω and p.

We require some preliminaries before stating our theorem for the radial domain. Consider the variational formulation of an eigenvalue problem given by

$$\mu_1 = \inf_{\psi \in H^1_{loc}(0,\frac{\pi}{4})} \Big\{ \int_0^{\frac{\pi}{4}} |\psi'(\theta)|^2 \omega(\theta)\, d\theta; \quad \int_0^{\frac{\pi}{4}} |\psi(\theta)|^2 \omega(\theta)\, d\theta = 1, \int_0^{\frac{\pi}{4}} \psi(\theta)\omega(\theta)\, d\theta = 0 \Big\}, \tag{80}$$

where $\omega(\theta) := \cos^{n-1}(\theta)\sin^{n-1}(\theta)$ and suppose ψ_1 satisfies the minimization problem. Then (μ_1, ψ_1) satisfies

$$\begin{cases} -\partial_\theta(\omega(\theta)\psi_1'(\theta)) = \mu_1 \omega(\theta)\psi_1(\theta) & \text{in } (0, \frac{\pi}{4}), \\ \psi'(\theta) > 0 & \text{in } (0, \frac{\pi}{4}), \\ \psi_1'(0) = \psi_1'(\frac{\pi}{4}) = 0, \end{cases} \tag{81}$$

and note (μ_1, ψ_1) is the second eigenpair, the first eigenpair is given by $(\mu_0, \psi_0) = (0, 1)$.
An easy computation shows that

$$\mu_1 = 4(N+2), \qquad \psi_1(\theta) = -\cos(4\theta) + \frac{2-N}{2+N}.$$

We also recall the definition of the best constant in Hardy inequality for the domain Ω, that is,

$$\beta_\lambda(\Omega) = \inf_{u \in H^1_0(\Omega)} \frac{\int_\Omega |\nabla u|^2\, dx + \lambda \int_\Omega u^2\, dx}{\int_\Omega \frac{u^2}{|x|^2}\, dx}. \tag{82}$$

We are now ready to state our general theorem regarding the existence of a non-radial solution for a fully radial problem.

Theorem 8.1. *Let u be the K_+ ground state solution obtained in either of Theorems 4.1, 5.1 or 6.1. If*

$$p - 2 > 4(N+2)/\beta_\lambda(\Omega),$$

then u is a nonradial function.

Proof. Let us assume that u is a radial function. Note that $K = K_+$ consists of functions $w = w(r, \theta)$ where $\theta \mapsto w(r, \theta)$ is non-decreasing on the interval $(0, \pi/4)$. Recall that $E_K(u) = c > 0$ where the critical value c is characterized by

$$c = \inf_{\gamma \in \Gamma} \max_{\tau \in [0,1]} E_K[\gamma(\tau)],$$

where

$$\Gamma = \{\gamma \in C([0,1], V) : \gamma(0) = 0 \neq \gamma(1), E_K(\gamma(1)) \leq 0\}.$$

For the sake of simplifying the notations, we use E instead of E_K in the rest of the proof. Let ψ_1 satisfies (81), and let ψ be the extension of ψ_1 evenly across $\theta = \frac{\pi}{4}$. Note that ψ solves the same equation on $(0, \frac{\pi}{2})$. Set $v(r, \theta) = u(r)\psi(\theta)$ and note that $u + tv$ belongs to the set K for $0 < t < 1$. We first show that

$$\int_\Omega |\nabla v|^2\, dx + \lambda \int_\Omega v^2\, dx - (p-1) \int_\Omega |a(|x|)u|^{p-2} v^2\, dx < 0. \tag{83}$$

To this end we need to show that $M(u,v)<0$ where

$$M(u,v) := \int_{\widehat{\Omega}} s^{n-1}t^{n-1}(v_t^2+v_s^2+\lambda v^2)\,ds\,dt - (p-1)\int_{\widehat{\Omega}} s^{m-1}t^{n-1}a(s,t)u^{p-2}v^2\,ds\,dt < 0. \tag{84}$$

Note first that it follows from the equation $-\Delta u+\lambda u = a(r)u^{p-1}$ that

$$\int_{R_1}^{R_2}(u_r^2+\lambda u^2)r^{N-1}\,dr = \int_{R_1}^{R_2} a(r)u^p r^{N-1}\,dr. \tag{85}$$

It also from the definition of $\beta=\beta_\lambda(\Omega)$, the best constant in Hardy inequality, that

$$\beta\int_{R_1}^{R_2}\frac{u^2}{r^2}r^{N-1}\,dr \le \int_{R_1}^{R_2}(u_r^2+\lambda u^2)r^{N-1}\,dr. \tag{86}$$

It follows from (85) by writing $M(u,v)$ in polar coordinates that

$$\begin{aligned} M(u,v) &= \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\Big(\psi^2u_r^2+\frac{u^2\psi'^2}{r^2}+\lambda u^2\psi^2-(p-1)a(r)u^p\psi^2\Big)r^{N-1}\omega(\theta)\,d\theta\,dr \\ &= \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\frac{u^2\psi'^2}{r^2}r^{N-1}\omega(\theta)\,d\theta\,dr-(p-2)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\psi^2(u_r^2+\lambda u^2)r^{N-1}\omega(\theta)\,d\theta\,dr, \end{aligned}$$

where $\omega(\theta)=\cos^{n-1}(\theta)\sin^{n-1}(\theta)$. This together with the definition of $\mu_1=4(N+2)$ in (80) and the inequality (86) imply that

$$\begin{aligned} M(u,v) &= \mu_1\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\frac{u^2\psi^2}{r^2}r^{N-1}\omega(\theta)\,d\theta\,dr-(p-2)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\psi^2(u_r^2+\lambda u^2)r^{N-1}\omega(\theta)\,d\theta\,dr \\ &= \int_0^{\frac{\pi}{2}}|\psi(\theta)|^2\omega(\theta)\,d\theta\Big(\mu_1\int_{R_1}^{R_2}\frac{u^2}{r^2}r^{N-1}\,dr-(p-2)\int_{R_1}^{R_2}(u_r^2+\lambda u^2)r^{N-1}\,dr\Big) \\ &\le \int_0^{\frac{\pi}{2}}|\psi(\theta)|^2\omega(\theta)\,d\theta\Big(\frac{\mu_1}{\beta}\int_{R_1}^{R_2}u_r^2r^{N-1}\,dr-(p-2)\int_{R_1}^{R_2}(u_r^2+\lambda u^2)r^{N-1}\,dr\Big) \\ &= \int_0^{\frac{\pi}{2}}|\psi(\theta)|^2\omega(\theta)\,d\theta\int_{R_1}^{R_2}(u_r^2+\lambda u^2)r^{N-1}\,dr\Big(\frac{\mu_1}{\beta}-(p-2)\Big)<0, \end{aligned}$$

where the last inequality follows from the fact that

$$\frac{\mu_1}{\beta}-(p-2)=\frac{4(N+2)}{\beta}-(p-2)<0.$$

Set $\gamma_\sigma(\tau)=\tau(u+\sigma v)l$, where $l>0$ is chosen in such a way that $E\big((u+\sigma v)l\big)\le 0$ for all $|\sigma|\le 1$. Note that $\gamma_\sigma\in\Gamma$. We shall show that there exists $\sigma>0$ such that for every $\tau\in[0,1]$ one has $E(\gamma_\sigma(\tau))<E(u)$, and therefore,

$$c\le\max_{\tau\in[0,1]}E(\gamma_\sigma(\tau))<E(u),$$

which leads to a contradiction since $E(u)=c$. Note first that there exists a unique smooth real function g on a small neighbourhood of zero with $g'(0)=0$ and $g(0)=1/l$ such that $\max_{\tau\in[0,1]}E(\gamma_\sigma(\tau))=E\big(g(\sigma)(u+\sigma v)l\big)$. We now define $h:\mathbb{R}\to\mathbb{R}$ by

$$h(\sigma)=E\big(g(\sigma)(u+\sigma v)l\big)-E(u).$$

Clearly we have $h(0)=0$. Note also that $h'(0)=0$ due to the facts that $E'(u)=0$ and $\int \psi\omega(\theta)\,d\theta = 0$. Finally $h''(0)<0$ due to (83). This in fact show that

$$\max_{\tau\in[0,1]} E(\gamma_\sigma(\tau)) = E\big(g(\sigma)(u+\sigma v)l\big) < E(u),$$

for small $\sigma>0$ as desired. $\square$

9 Domains of triple revolution

In this section we consider domains of triple revolution. In particular we consider

$$\begin{cases} -\Delta u = a(x)u^{p-1} & \text{in } \Omega, \\ u=0 & \text{on } \partial\Omega, \end{cases} \tag{87}$$

where Ω is a bounded domain in $\mathbb{R}^N$ which has a smooth boundary and which is a domain of triple revolution. Consider

$$s=\left\{x_1^2+\cdots+x_m^2\right\}^{\frac{1}{2}}, \quad t=\left\{x_{m+1}^2+\cdots+x_{m+n}^2\right\}^{\frac{1}{2}}, \quad \tau:=\left\{x_{m+n+1}^2+\cdots+x_N^2\right\}^{\frac{1}{2}},$$

so s,t,τ has dimension $m,n,l=N-(m+n)$ respectively. Here the function a is a function of (t,s,τ), that is $a=a(t,s,\tau)$.

Remark 9.1. *Note that a radial domain and a domain of double revolution are particular cases of domains of triple revolution. However, domains of triple revolutions are not necessarily radial or domains of double revolution. Besides providing a framework to deal with more general domains, this will create a pathway to prove several multiplicity results for positive solutions on radial domains. For instance an annulus can be seen as a radial domain and a domain of double revolution as well as a domain of triple revolution. Thus, one can obtain new positive solutions for a radial problem by looking into solutions having a nontrivial triple symmetry. This is indeed the main motivation for this section.*

In the previous sections we used polar coordinates in the (s,t) plane. In this section we will use spherical coordinates to describe the coordinates (s,t,τ):

$$s=r\sin(\theta)\cos(\varphi), \quad t=r\sin(\theta)\sin(\varphi), \quad \tau=r\cos(\theta), \tag{88}$$

where $0<\theta<\pi$, $0<\varphi<2\pi$ and $r>0$; but of course we have restricted (s,t,τ) to the first octant in $\mathbb{R}^3$ and hence $0<\theta<\frac{\pi}{2}$, $0<\varphi<\frac{\pi}{2}$, and $r>0$. Note that the function a can be also seen as a function of (φ,θ,r), that is $a=a(\varphi,\theta,r)$.

The monotonicity we will use will be in φ and hence it is also very natural to consider cylindrical coordinates for (s,t,τ) but we chose spherical for variety and also since we have the case of an annulus in mind which may be more natural to consider spherical coordinates.

We now define

$$U=\left\{(s,t,\tau)\in\mathbb{R}^3 : x=(x_1=s, x_2=0,...,x_m=0, x_{m+1}=t, x_{m+2}=0,...,x_N=\tau)\in\Omega\right\},$$

where $x_i=0$ for $i\notin\{1,n+1,N\}$. We define $\widehat{\Omega}=\{(s,t,\tau)\in U : s,t,\tau>0\}$. We now define

$$\widetilde{\Omega}=\left\{(\varphi,\theta,r)\in\left(0,\frac{\pi}{2}\right)\times\left(0,\frac{\pi}{2}\right)\times(0,\infty) : (s,t,\tau)\in\widehat{\Omega}\right\},$$

and we also define a subset of $\widetilde{\Omega}$ given by

$$\widetilde{\Omega}_0 = \left\{ (\varphi, \theta, r) \in \left(0, \frac{\pi}{4}\right) \times \left(0, \frac{\pi}{2}\right) \times (0, \infty) : (s, t, \tau) \in \widehat{\Omega} \right\},$$

where note the only change is we are now restricting $0 < \varphi < \frac{\pi}{4}$.
Take $G = O(m) \times O(n) \times O(l)$ and consider

$$H^1_{0,G}(\Omega) = \left\{ u \in H^1_0(\Omega) : gu = u \ \ \forall g \in G \right\}.$$

We are now ready to state our monotonicity assumptions for the domians of triple revolution.

Definition 9.2. *[**The monotonicity assumption on the functions and the domain**] Let Ω be a bounded domain of triple revoluion in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$.*

1. *(K_- definition and domain assumptions) Suppose $g^i = g^i(\varphi, \theta)$ is smooth and positive on $[0, \frac{\pi}{2}] \times [0, \pi/2]$ and for each fixed $\theta \in (0, \pi/2)$ we have: $\varphi \mapsto g^i(\varphi, \theta)$ even about $\varphi = \frac{\pi}{4}$, for $i = 2$ we have the map is decreasing in φ on $(0, \frac{\pi}{4})$ and $i = 1$ we have it increasing in φ on $(0, \frac{\pi}{4})$. We also $g^1 < g^2$ on $[0, \frac{\pi}{2}] \times [0, \pi/2]$. We consider domains where*

   $$\widetilde{\Omega} = \left\{ (\varphi, \theta, r) : g^1(\varphi, \theta) < r < g^2(\varphi, \theta) \text{ for } (\varphi, \theta) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \right\}.$$

   Define K_- to be the set of nonnegative functions $u \in H^1_{0,G}(\Omega)$ with $u_\varphi \leq 0$ in $\widetilde{\Omega}_0$ and which are even across $\theta = \frac{\pi}{4}$.

2. *(K_+ definition and domain assumptions) Suppose $g^i = g^i(\varphi, \theta)$ is smooth and positive on $[0, \frac{\pi}{2}] \times [0, \pi/2]$ and for each fixed $\theta \in (0, \pi/2)$ and $i = 1, 2$ we have $\varphi \mapsto g^i(\varphi, \theta)$ is constant on $(0, \pi/2)$. We consider domains Ω where*

   $$\widetilde{\Omega} = \left\{ (\varphi, \theta, r) : g^1(\varphi, \theta) < r < g^2(\varphi, \theta) \text{ for } (\varphi, \theta) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \right\}.$$

   Note this includes the case of an annulus. Define K_+ to be the set of nonnegative functions $u \in H^1_{0,G}(\Omega)$ with $u_\varphi \geq 0$ in $\widetilde{\Omega}_0$ and which are even across $\varphi = \frac{\pi}{4}$.

3. *($K_{-,\frac{\pi}{2}}$ definition and domain assumptions) Suppose $g^i = g^i(\varphi, \theta)$ is smooth and positive on $[0, \pi/2] \times [0, \pi/2]$ and for each fixed $\theta \in (0, \pi/2)$ we have: the map $\varphi \mapsto g^2(\varphi, \theta)$ is decreasing in φ on $(0, \frac{\pi}{2})$ and $\varphi \mapsto g^1(\varphi, \theta)$ is increasing in φ on $(0, \frac{\pi}{2})$. We alsohave $g^1 < g^2$ on $[0, \frac{\pi}{2}] \times [0, \pi/2]$. We consider domains Ω where*

   $$\widetilde{\Omega} = \left\{ (\varphi, \theta, r) : g^1(\varphi, \theta) < r < g^2(\varphi, \theta) \text{ for } (\varphi, \theta) \in \left(0, \frac{\pi}{2}\right) \times \left(0, \frac{\pi}{2}\right) \right\}.$$

   Define $K_{-,\frac{\pi}{2}}$ to be the set of nonnegative functions $u \in H^1_{0,G}(\Omega)$ with $u_\varphi \leq 0$ in $\widetilde{\Omega}$.

Here we state our main theorem for this section.

Theorem 9.1. *Let Ω be a bounded domain of triple revolution in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$ and consider (87) with $a = a(\varphi, \theta, r)$ positive and sufficiently smooth.*

1. *Suppose $m = n$ and Ω is a domain satisfying the symmetry condition part 1 of Definition 9.2 and $a_\varphi \leq 0$ in $\widetilde{\Omega}_0$. Then for all*

   $$2 < p < \min \left\{ \frac{2(n+m+1)}{n+m-1}, \frac{2(n+l+1)}{n+l-1} \right\},$$

   there is a positive classical K_- ground state solution u of (87). Note this case includes the case of Ω an annulus.

2. *Suppose $m = n$ and Ω is a domain satisfying the symmetry condition part 2 on Definition 9.2 and $a_\varphi \geq 0$ in $\widetilde{\Omega}_0$. Then for all*

$$2 < p < \min\left\{\frac{2(l+2)}{l}, \frac{2(n+m+1)}{n+m-1}\right\},$$

there is a positive classical K_+ ground state solution u of (87). Note this case includes the case of Ω an annulus.

3. *Suppose Ω is a domain satisfying the symmetry condition part 3 on Definition 9.2 with $n \leq m$ and $a_\varphi \leq 0$ in $\widetilde{\Omega}$. Then for all*

$$2 < p < \min\left\{\frac{2(n+m+1)}{n+m-1}, \frac{2(n+l+1)}{n+l-1}\right\},$$

there is a positive classical $K_{-,\frac{\pi}{2}}$ ground state solution u of (87).

Before discussing the proofs we write out some formula's we will need soon. Given a function $v(x)$ defined on Ω (which has the G symmetry) we have

$$\int_\Omega v(x)dx = c\int_{\widehat{\Omega}} v(s,t,\tau)s^{n-1}t^{n-1}\tau^{l-1}dsdtd\tau,$$

where we are abusing notation as usual. If we further abuse notation we can write this in terms of spherical coordinates as

$$\int_{\widetilde{\Omega}} v(\varphi,\theta,r)d\mu(\varphi,\theta,r)$$

where

$$d\mu(\varphi,\theta,r) = r^{N-1}\sin^{m+n-1}(\theta)\cos^{m-1}(\varphi)\sin^{n-1}(\varphi)\cos^{l-1}(\theta)d\varphi d\theta dr,$$

and in the case of $m = n$ we have

$$d\mu(\varphi,\theta,r) = r^{2n+l-1}\sin^{2n-1}(\theta)\cos^{n-1}(\varphi)\sin^{n-1}(\varphi)\cos^{l-1}(\theta)d\varphi d\theta dr.$$

Also note we can write the square of the gradient as

$$|\nabla u(x)|^2 = u_r^2 + \frac{u_\theta^2}{r^2} + \frac{u_\varphi^2}{r^2\sin^2(\theta)}.$$

As before we begin by examining the added compactness one gets.

Theorem 9.2. *(Imbeddings for annular domains) Let Ω denote an annular of triple revolution in $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n \times \mathbb{R}^l$.*

1. *(Imbedding without monotonicity) Suppose Ω has no monotonicity and*

$$1 \leq p < p_1(m,n,l) := \min\left\{\frac{2(n+m+1)}{n+m-1}, \frac{2(m+l+1)}{m+l-1}, \frac{2(n+l+1)}{n+l-1}\right\}.$$

Then $H^1_{0,G}(\Omega) \subset\subset L^p(\Omega)$.

2. *(Imbedding with monotonicity) Suppose Ω satisfy the symmetry condition part 1 in Definition 9.2, $n \leq m$ and*

$$1 \leq p < p_2(m,n,l) := \min\left\{ \frac{2(n+m+1)}{n+m-1}, \frac{2(n+l+1)}{n+l-1} \right\}.$$

Then $K_- \subset\subset L^p(\Omega)$.

3. *Suppose Ω satisfy the symmetry condition part 3 in Definition 9.2, $n \leq m$ and*

$$1 \leq p < p_2(m,n,l) := \min\left\{ \frac{2(n+m+1)}{n+m-1}, \frac{2(n+l+1)}{n+l-1} \right\}.$$

Then $K_{-,\frac{\pi}{2}} \subset\subset L^p(\Omega)$.

4. *Suppose Ω satisfy the symmetry condition part 2 in Definition 9.2 and*

$$1 \leq p < p_3(m,n,l) := \min\left\{ \frac{2(l+2)}{l}, \frac{2(n+m+1)}{n+m-1} \right\}.$$

Then $K_+ \subset\subset L^p(\Omega)$.

Proof. 1. This part follows from Theorem 3.1.

2. By using spherical coordinates for (s,t,τ)

$$s = r\sin(\theta)\cos(\varphi), \quad t = r\sin(\theta)\sin(\varphi), \quad \tau = r\cos(\theta),$$

we have that

$$\int_{\widehat{\Omega}} u(s,t,\tau)^p s^{m-1} t^{n-1} \tau^{l-1} ds dt d\tau$$
$$= \int_0^{\pi/2}\int_0^{\pi/2}\int_{g_1}^{g_2} r^{N-1}\sin^{m-1}(\theta)\cos^{m-1}(\varphi)\sin^{n-1}(\theta)\sin^{n-1}(\varphi)\cos^{l-1}(\theta)u(\varphi,\theta,r)^p\, dr\, d\theta d\varphi.$$

For $\varphi \in [\pi/3, \pi/2]$ we have that $\sin(\varphi) \leq c\sin(\varphi - \pi/4)$ for some constant $c > 0$. Thus, considering the evenness properties of g_1, g_2 and $\varphi \mapsto u(\varphi,\theta,r)$ across $\varphi = \frac{\pi}{4}$ we obtain that

$$\int_{\pi/3}^{\pi/2}\int_{g_1(\varphi,\theta)}^{g_2(\varphi,\theta)} r^{N-1}\cos^{m-1}(\varphi)\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\, dr\, d\varphi$$
$$\leq c^{n-1}\int_{\pi/3}^{\pi/2}\int_{g_1(\varphi-\pi/4,\theta)}^{g_2(\varphi-\pi/4,\theta)} r^{N-1}\cos^{m-1}(\varphi-\pi/4)\sin^{n-1}(\varphi-\pi/4)u(\varphi-\pi/4,\theta,r)^p\, dr\, d\varphi$$
$$= c^{n-1}\int_{\pi/12}^{\pi/4}\int_{g_1(\varphi,\theta)}^{g_2(\varphi,\theta)} r^{N-1}\cos^{m-1}(\varphi)\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\, dr\, d\varphi.$$

Thus, there is a constant $C_1 > 0$ such that

$$\int_0^{\pi/2}\int_{g_1}^{g_2} r^{N-1}\cos^{m-1}(\varphi)\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\, dr\, d\varphi$$
$$\leq C_1\int_0^{\pi/3}\int_{g_1}^{g_2} r^{N-1}\cos^{m-1}(\varphi)\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\, dr\, d\varphi.$$

On the other hand,

$$\int_{\pi/4}^{\pi/2}\int_0^{\pi/3}\int_{g_1}^{g_2} r^{N-1}\cos^{m-1}(\varphi)r^{n-1}\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\sin^{n+m-2}(\theta)\cos^{l-1}(\theta)\,dr\,d\varphi\,d\theta$$

$$= \int_{\{\widehat{\Omega},\ s\geq\beta\}} u(s,t,\tau)^p s^{m-1}t^{n-1}\tau^{l-1}dsdtd\tau \qquad (89)$$

and

$$\int_0^{\pi/4}\int_0^{\pi/3}\int_{g_1}^{g_2} r^{N-1}\cos^{m-1}(\varphi)r^{n-1}\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\sin^{n+m-2}(\theta)\cos^{l-1}(\theta)\,dr\,d\varphi\,d\theta$$

$$= \int_{\{\widehat{\Omega},\ \tau\geq\beta\}} u(s,t,\tau)^p s^{m-1}t^{n-1}\tau^{l-1}dsdtd\tau \qquad (90)$$

for some positive constant β. Therefore, for (89), we have

$$\left(\int_{\{\widehat{\Omega},\ s\geq\beta\}} u(s,t,\tau)^p s^{m-1}t^{n-1}\tau^{l-1}dsdt\right)^{2/p} \leq C_2\left(\int_{\{\widehat{\Omega},\ s\geq\beta\}} u(s,t,\tau)^p t^{n-1}\tau^{l-1}dsdt\right)^{2/p}.$$

Thus, by part 1),

$$\begin{aligned}\left(\int_{\{\widehat{\Omega},\ s\geq\beta\}} u(s,t,\tau)^p t^{n-1}\tau^{l-1}dsdt\right)^{2/p} &\leq C_3\int_{\{\widehat{\Omega},\ s\geq\beta\}}(u^2+u_s^2+u_t^2+u_\tau^2)t^{n-1}\tau^{l-1}dsdtd\tau\\ &\leq C_4\int_{\{\widehat{\Omega},\ s\geq\beta\}}(u^2+u_s^2+u_t^2+u_\tau^2)t^{n-1}s^{m-1}\tau^{l-1}dsdtd\tau\\ &\leq C_4\int_{\widehat{\Omega}}(u^2+u_s^2+u_t^2+u_\tau^2)t^{n-1}s^{m-1}\tau^{l-1}dsdtd\tau = C_5\|u\|^2_{H^1(\Omega)}.\end{aligned}$$

By a similar argument for (90) we have

$$\int_0^{\pi/4}\int_0^{\pi/3}\int_{g_1}^{g_2} r^{N-1}\cos^{m-1}(\varphi)r^{n-1}\sin^{n-1}(\varphi)u(\varphi,\theta,r)^p\,dr\,d\varphi\,d\theta \leq C_6\|u\|^p_{H^1(\Omega)},$$

from which the desired result follows.

3. Proof follows by the same argument as in part 2.
4. Proof follows by the same argument as in the proof of Theorem 3.1.

□

Remark 9.3. *It is worth noting that $p_i(m,n,l)$ for $i=1,2,3$ in Theorem 9.2 give an improved embedding beyond the standard Sobolev embeddings. In fact, we have the following,*

- $p_1(m,n,l) > \frac{2N}{N-2}$ *provided* $m,n,l>1$.
- $p_2(m,n,l) \geq p_1(m,n,l)$, *provided* $m\geq n$.
- $p_3(m,n,l) > \frac{2N}{N-2}$ *if and only if* $1<l<N-2$. *Also,* $p_3(m,n,l)\geq p_i(m,n,l)$ *for* $i=1,2$ *provided* $n,m>1$.

Moreover, if N is odd then $p_3(m,n,l)$ is maximized (here N is fixed and we are varying m,n,l) when $l=(N-1)/2$ with the value

$$p_3(m,n,\frac{N-1}{2})=\frac{2(N+3)}{N-1}$$

and note that

$$\frac{2(N+3)}{N-1}>\frac{2N}{N-2} \text{ if and only if } N>3.$$

As before we consider the following linear problem given by

$$\begin{cases} -\Delta v = a(x)u^{p-1} & \text{in } \Omega, \\ v=0 & \text{on } \partial\Omega. \end{cases} \tag{91}$$

Theorem 9.3. *(Pointwise invariance property)*

1. *Suppose $m=n\geq 1$, Ω satisfies the K_- domain assumptions from Definition 9.2 and $a_\varphi \leq 0$ in $\widetilde{\Omega}_0$. If $u\in K_-$ and v satisfies (91) then $v\in K_-$.*

2. *Suppose $m=n\geq 1$, Ω satisfies the K_+ domain assumptions from Definition 9.2 and $a_\varphi \geq 0$ in $\widetilde{\Omega}_0$. If $u\in K_+$ and v satisfies (91) then $v\in K_+$.*

3. *Suppose Ω satisfies the $K_{-,\frac{\pi}{2}}$ domain assumptions from Definition 9.2 and $a_\varphi\leq 0$ in $\widetilde{\Omega}$. If $u\in K_{-,\frac{\pi}{2}}$ and v satisfies (91) then $v\in K_{-,\frac{\pi}{2}}$.*

Remark 9.4. *One can surely remove the $n\geq 2$ restriction but when proving $w=0$ in Ω one needs to try a bit harder when choosing a suitable cut off (here we would have $dim(\Gamma)=N-2$ and not strictly less that $N-2$).*

Proof of Theorem 9.3. Parts 1,2: We begin by taking $u\in K_\pm$ since much of the proof is the same for either case and as before we consider $u_k=\min\{u(x),k\}$ where k is a large integer and note $u_k\in K_\pm$. Let v^k denote a solution of (91) with u replaced with u_k and then note by elliptic regularity we have $v^k\in H^1_{0,G}(\Omega)\cap C^{1,\delta}(\overline{\Omega})$ for all $0<\delta<1$. Now note we can write

$$\Delta v^k(x)=v^k_{ss}+v^k_{tt}+v^k_{\tau\tau}+\frac{(m-1)v^k_s}{s}+\frac{(n-1)v^k_t}{t}+\frac{(l-1)v^k_\tau}{\tau},$$

and a computation shows that

$$\frac{v^k_s}{s}+\frac{v^k_t}{t} = \frac{2v^k_r}{r}+\frac{2v^k_\theta}{r^2\tan(\theta)}+\frac{v^k_\varphi}{r^2\sin^2(\theta)}\left(\frac{1}{\tan(\varphi)}-\tan(\varphi)\right).$$

and

$$\frac{v^k_\tau}{\tau}=\frac{v^k_r}{r}-\frac{\tan(\theta)v^k_\theta}{r^2}.$$

From this we see that the equation for v^k in spherical coordinates is given by $L(v^k)=au_k^{p-1}$ where

$$\begin{aligned} L(v) &= \left\{-v_{rr}-\frac{(2n+l-1)v_r}{r}-\frac{v_{\varphi\varphi}}{r^2\sin^2(\theta)}-\frac{v_{\theta\theta}}{r^2}-\frac{v_\theta}{r^2}\left(\frac{2n-1}{\tan(\theta)}-(l-1)\tan(\theta)\right)\right\} \\ &\quad +\frac{(n-1)v_\varphi}{r^2\sin^2(\theta)}h(\varphi), \\ &= L_0(v)+\frac{(n-1)v_\varphi}{r^2\sin^2(\theta)}h(\varphi), \end{aligned}$$

where $h(\varphi) = \tan(\varphi) - \frac{1}{\tan(\varphi)}$. Note from this we see that

$$\partial_\varphi L(v) = L(v_\varphi) + \frac{(n-1)v_\varphi}{r^2 \sin^2(\theta)} h'(\varphi). \tag{92}$$

We now show that v^k has the desired symmetry across $\varphi = \frac{\pi}{4}$. Note we have

$$L(v^k)(\varphi,\theta,r) = a(\varphi,\theta,r) u_k(\varphi,\theta,r)^{p-1} =: g(\varphi,\theta,r) \quad \text{ in } \widetilde{\Omega},$$

with suitable boundary conditions. Define

$$\widehat{v}(\varphi,\theta,r) = v^k\left(\frac{\pi}{2} - \varphi, \theta, r\right),$$

and hence our goal is to show that $\widehat{v} = v$ which would prove v^k is even in φ across $\varphi = \frac{\pi}{4}$. A computation shows

$$L(\widehat{v})(\varphi,\theta,r) = L_0(v^k)(\frac{\pi}{2} - \varphi,\theta,r) + \frac{(n-1)(-1)v_\varphi(\frac{\pi}{2} - \varphi,\theta,r)}{r^2 \sin^2(\theta)} h(\varphi)$$

but noting that h is odd across $\varphi = \frac{\pi}{4}$, ie. $-h(\varphi) = h(\frac{\pi}{2} - \varphi)$, we have

$$L(\widehat{v})(\varphi,\theta,r) = L(v^k)(\frac{\pi}{2} - \varphi,\theta,r) = g(\frac{\pi}{2} - \varphi,\theta,r) = g(\varphi,\theta,r)$$

after noting that g is even across $\varphi = \frac{\pi}{4}$ since both a and u_k are. Hence we see that $L(\widehat{v})(\varphi,\theta,r) = L(v^k)(\varphi,\theta,r)$ in $\widetilde{\Omega}$. We now discuss the boundary conditions for v^k (and $\widehat{v}$) in some detail. This will be more needed later when we examine the monotonicity of v^k. Define

$$\Gamma_{\varphi=0} = \left\{(\varphi = 0,\theta,r) : g^1(0,\theta) < r < g^2(0,\theta), 0 < \theta < \frac{\pi}{2}\right\} \text{ and similarly } \Gamma_{\varphi=\frac{\pi}{2}}, \tag{93}$$

$$\Gamma_{r=g^1} = \left\{(\varphi,\theta,g^1(\varphi,\theta)) : 0 < \varphi < \frac{\pi}{2}, 0 < \theta < \frac{\pi}{2}\right\} \text{ and similarly } \Gamma_{r=g^2}, \tag{94}$$

$$\Gamma_{\theta=0} = \left\{(\varphi,\theta = 0,r) : g^1(\varphi,0) < r < g^2(\varphi,0), 0 < \varphi < \frac{\pi}{2}\right\} \text{ and similarly } \Gamma_{\theta=\frac{\pi}{2}}. \tag{95}$$

First note that $v^k, \widehat{v}$ are both zero on $\Gamma_{r=g^i}$ for $i = 1,2$ (to see the result for $\widehat{v}$ use the fact that g^i is even across $\varphi = \frac{\pi}{4}$. By the smoothness of v^k (and hence $\widehat{v}$) (and since the functions are even across $\varphi = 0$ and $\varphi = \frac{\pi}{2}$) we have $v^k_\varphi = \widehat{v}_\varphi = 0$ on $\Gamma_{\varphi=0}$ and $\Gamma_{\varphi=\frac{\pi}{2}}$. By smoothness and symmetry we also get $v^k_\theta = \widehat{v}_\theta = 0$ on $\Gamma_{\theta=\frac{\pi}{2}}$. Note $\Gamma_{\theta=0}$ corresponds to a portion of the positive τ axis. Set $w(\varphi,\theta,r) = v^k(\varphi,\theta,r) - \widehat{v}(\varphi,\theta,r)$ defined on $\widetilde{\Omega}$. Also note we have $L(w)(\varphi,\theta,r) = 0$ for $(\varphi,\theta,r) \in \widetilde{\Omega}$ with $w = 0$ on $\Gamma_{r=g^i}$ for $i = 1,2$; $w_\varphi = 0$ on $\Gamma_{\varphi=0} \cup \Gamma_{\varphi=\frac{\pi}{2}}$ and $w_\theta = 0$ on $\Gamma_{\theta=\frac{\pi}{2}}$. Set $\Gamma = \{x \in \Omega : s = t = 0\}$ and note that $dim(\Gamma) = l = N - 2n \le N - 2$. Also note in terms of x we have

$$\Delta w(x) = 0 \quad \text{ in } \Omega \backslash \Gamma,$$

with $w = 0$ on $\partial\Omega$. We now claim that since $w \in C^{1,\alpha}(\overline{\Omega}) \cap C^\infty(\overline{\Omega} \backslash \Gamma)$ and since $dim(\Gamma) \le N - 2$ we have $\Delta w = 0$ in Ω in sense of distributions and then we can apply the maximum principle to see $w = 0$ in Ω.

We now prove the claim. Take a smooth function g on $\mathbb{R}$ with $g(t) = 0$ for $t \leq 1$ and $g(t) = 1$ for $t \geq 2$ and consider $\delta_\Gamma(x) = dist(x, \Gamma)$ (the Euclidean distance) and fix $x_0 \in \Gamma$ but not an endpoint since the endpoints lie on $\partial\Omega$. Note that δ_Γ is smooth near x_0 and we now set

$$\gamma_\varepsilon(x) = g\left(\frac{\delta_\Gamma(x)}{\varepsilon}\right),$$

and note g_ε is smooth near x_0. Let ψ be smooth and compactly supported near x_0 and note a computation shows that

$$\begin{aligned} \left| \int_\Omega \gamma_\varepsilon w \Delta\psi dx \right| &= \left| \nabla\gamma_\varepsilon \cdot \{\nabla w \psi - w \nabla\psi\} dx \right| \\ &\leq C \int_\Omega |\nabla\gamma_\varepsilon(x)| dx \end{aligned}$$

where C independent of ε for small ε. We now claim the right hand side converges to zero and hence we'd have $\int_\Omega w\Delta\psi dx = 0$ which shows that $\Delta w = 0$ in Ω in the sense of distributions. We can now use Hausdorff measure to prove the result but we prefer to use the box counting dimension, see [23] for instance. Note that we have

$$N - 2 \geq dim_{box}(\Gamma) := N - \lim_{t \searrow 0} \frac{\log(|\Gamma_t|)}{\log(t)},$$

where $|\Gamma_t|$ is the N dimensional measure of $\Gamma_t = \{x \in \Omega : \delta_\Gamma(x) < t\}$. So there is some $\alpha(t) \to 0$ as $t \searrow 0$ such that $|\Gamma_t| \leq t^{\alpha(t)+2}$. Then note we have

$$\begin{aligned} \int_\Omega |\nabla\gamma_\varepsilon(x)| dx &\leq C \int_{\varepsilon < \delta_\Gamma < 2\varepsilon} \frac{1}{\varepsilon} dx \\ &\leq C \frac{|\Gamma_{2\varepsilon}|}{\varepsilon} \\ &\leq C \frac{(2\varepsilon)^{2+\alpha(2\varepsilon)}}{\varepsilon} \to 0, \end{aligned}$$

which proves the claim.

Monotonicity. We now show that v^k has the desired monotonicity in φ on $\widetilde{\Omega}_0$. Note that by (92) we see

$$L(v^k_\varphi) + \frac{(n-1)h'(\varphi)v^k_\varphi}{r^2 \sin^2(\theta)} = \partial_\varphi(a u_k^{p-1}) \quad \text{in } \widetilde{\Omega}_0, \tag{96}$$

and note $h'(\varphi) \geq 0$ and hence there is hope for a maximum principle for the operator on the left acting on v^k_φ. We now define the boundaries and note we are really taking the boundaries from above and suitably adjusting them to $0 < \varphi < \frac{\pi}{4}$ instead of $0 < \varphi < \frac{\pi}{2}$. So we have

$$\Gamma^0_{\varphi=0} = \left\{ (\varphi = 0, \theta, r) : g^1(0,\theta) < r < g^2(0,\theta), 0 < \theta < \frac{\pi}{2} \right\} \text{ and similarly } \Gamma^0_{\varphi=\frac{\pi}{4}}, \tag{97}$$

$$\Gamma^0_{r=g^1} = \left\{ (\varphi, \theta, g^1(\varphi,\theta)) : 0 < \varphi < \frac{\pi}{4}, 0 < \theta < \frac{\pi}{2} \right\} \text{ and similarly } \Gamma^0_{r=g^2}, \tag{98}$$

$$\Gamma^0_{\theta=0} = \left\{ (\varphi, \theta = 0, r) : g^1(\varphi, 0) < r < g^2(\varphi, 0), 0 < \varphi < \frac{\pi}{4} \right\} \text{ and similarly } \Gamma^0_{\theta=\frac{\pi}{2}}, \tag{99}$$

Boundary terms $\Gamma^0_{\varphi=0}\cup\Gamma^0_{\varphi=\frac{\pi}{4}}$. Note by the smoothness and symmetry of v^k we have $v^k_\varphi=0$ on $\Gamma^0_{\varphi=0}\cup\Gamma^0_{\varphi=\frac{\pi}{4}}$.

Boundary terms $\Gamma^0_{r=g^1}\cup\Gamma^0_{r=g^2}$. The boundary conditions here depend on with case of $K_\pm$ we are in. First consider the case of K_+. In this case because $v^k=0$ on $\Gamma^0_{r=g^1}\cup\Gamma^0_{r=g^2}$ and g^i is constant in φ we see that $v^k_\varphi=0$ on $\Gamma^0_{r=g^1}\cup\Gamma^0_{r=g^2}$. We now suppose we in the case of K_-. In this case we are either in the case of a annulus or a more general domain with suitable monotonicity. In the case of a annulus we have $v^k_\varphi=0$ on $\Gamma^0_{r=g^1}\cup\Gamma^0_{r=g^2}$ as in the case of K_+. Using the fact that $v^k\ge 0$ in Ω with $v^k=0$ on $\partial\Omega$ and the monotonicity of the maps $\varphi\mapsto g^i(\varphi,\theta)$ we see that $v^k_\varphi\le 0$ on $\Gamma^0_{r=g^1}\cup\Gamma^0_{r=g^2}$.

Boundary terms $\Gamma^0_{\theta=0}\cup\Gamma^0_{\theta=\frac{\pi}{2}}$. First we consider $\Gamma^0_{\theta=\frac{\pi}{2}}$. Note by the smoothness of v^k we have $v^k_\theta=0$ on $\Gamma^0_{\theta=\frac{\pi}{2}}$ and hence we have $0=(v^k_\theta)_\psi=(v^k_\psi)_\theta$ on $\Gamma^0_{\theta=\frac{\pi}{2}}$. We now examine the term $\Gamma^0_{\theta=0}$. Note that we can write $\nabla v^k(x)$ as

$$\nabla v^k(x)=v^k_r\hat{r}+\frac{v^k_\theta}{r}\hat{\theta}+\frac{v^k_\varphi}{r\sin(\theta)}\hat{\varphi},\tag{100}$$

where $(\hat{\varphi},\hat{\theta},\hat{r})$ are the unit vectors in spherical coordinates. From this we see that

$$|v^k_\varphi|\le r\sin(\theta)|\nabla v^k(x)|.\tag{101}$$

This shows that, at least in some limiting sense, we have $v^k_\varphi=0$ on $\Gamma^0_{\theta=0}$. We can now either work in spherical coordinates or translate back to coordinates in x; we will choose the latter since its more familiar to apply the maximum principle. Writing the left hand side of (96) we arrive at

$$-\Delta v^k_\varphi(x)+(n-1)H(x)v^k_\varphi(x)=\partial_\varphi(au_k^{p-1}),\qquad \text{in }\Omega_0,$$

where $\Omega_0:=\{x\in\Omega:0<\varphi<\frac{\pi}{4},x\notin\Gamma\}$ and

$$H(x)=\frac{\sum_{i=1}^{2n}x_i^2}{\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=n+1}^{2n}x_i^2\right)}.$$

We now consider the case of K_-. Let $\varepsilon>0$ be small and consider $\psi=(v^k_\varphi-\varepsilon)_+$ and note $\psi=0$ near Γ after considering (101) and also $\psi=0$ near the portions of the boundary of Ω_0 corresponding to $\varphi=0$ and $\varphi=\frac{\pi}{4}$ (but we really will only need the result for $\varphi=0$ since H is not singular at $\varphi=\frac{\pi}{4}$). Note that $\psi=0$ near $\partial\Omega_0$. From this we have

$$\int_{\Omega_0}\nabla v^k_\varphi\cdot\nabla\psi dx+(n-1)\int_{\Omega_0}Hv^k_\varphi\psi dx=\int_{\Omega_0}\partial_\varphi(au_k^{p-1})\psi dx\le 0,$$

after noting the assumptions on a and u. From this one sees that

$$\int_{\Omega_0}|\nabla\psi|^2dx+(n-1)\int_{\Omega_0}H\psi^2dx\le 0,$$

and hence we have $\psi = 0$ in Ω_0 and hence we have $v_\varphi^k \le \varepsilon$ a.e. in Ω_0 and hence we have the desired result after noting $\varepsilon > 0$ is arbitrary.

We now consider the case of K_+. Consider $\psi = (v_\varphi^k + \varepsilon)_-$ where $\varepsilon > 0$ is small. Then note we have $\psi = 0$ near $\partial\Omega_0$. As above we get

$$\int_{\Omega_0} \nabla v_\varphi^k \cdot \nabla \psi dx + (n-1) \int_{\Omega_0} H v_\varphi^k \psi dx = \int_{\Omega_0} \partial_\varphi (a u_k^{p-1}) \psi dx \ge 0,$$

after noting the assumptions on a and u. From this we can argue that

$$\int_{\Omega_0} |\nabla \psi|^2 dx + (n-1) \int_{\Omega_0} H \psi^2 dx \le 0,$$

and hence $\psi = 0$ which gives $v_\varphi^k \ge -\varepsilon$ and hence we get the desired result. We now need to pass to the limit in k, but this follows from similiar arguments that we used in previous sections.

3. The proof for this part follows from similar type computations as in [26] and some of the ideas used in part 1 and 2 of the previous proof to deal with the extra variable τ, we omit the details. □

Proof of Theorem 9.1. Once again, we are going to use Theorem 2.1 for the proof. Note that conditions (i) and (ii) in Theorem 2.1 follows from Theorems 9.2 and 9.3 respectively. This proves the existence of a non-negative weak solution u of (87). To prove the solution is positive and regular we use the same arguments we have used in the previous sections, we omit the details.

□

9.1 Nonsymmetric solutions on domains of triple revolution

In this section we examine the case where the domain, the equation and a have added symmetry and we examine the existence of solutions which do not inherit the same symmetry. We also recall the definition of the best constant in Hardy inequality for the domain Ω, that is,

$$\beta_0(\Omega) = \inf_{u \in H_0^1(\Omega)} \frac{\int_\Omega |\nabla u|^2 \, dx}{\int_\Omega \frac{u^2}{|x|^2} \, dx}. \tag{102}$$

We first consider the case of radial symmetry and then we consider the case of cylindrical symmetry around the τ axis in the case of the variables (s, t, τ).

9.1.1 The case of the annulus

Here we examine the case of $a(x) = a(|x|)$ is radial, and Ω is the annulus $\Omega = \{x : R_1 \le |x| < R_2\}$.

Theorem 9.4. *Let u be the solution obtained in either parts of Theorem 9.1. If β_0 is large enough then u depends on all three variables in a non-trivial way.*

Proof. We just do the proof for part 3 of Theorem 9.1. Other cases follows by the same argument.
Define

$$w_l(\theta) = \sin^{N-l-1}(\theta) \cos^{l-1}(\theta), \qquad w_{m,n}(\varphi) = \cos^{m-1}(\varphi) \sin^{n-1}(\varphi).$$

Consider the variational formulation of eigenvalue problems given by

$$\mu_l = \inf_{\psi \in H^1_{loc}(0, \frac{\pi}{2})} \Big\{ \int_0^{\frac{\pi}{2}} |\psi'(\theta)|^2 w_l(\theta) \, d\theta; \quad \int_0^{\frac{\pi}{2}} |\psi(\theta)|^2 w_l(\theta) \, d\theta = 1, \int_0^{\frac{\pi}{2}} \psi(\theta) w_l(\theta) \, d\theta = 0 \Big\}, \tag{103}$$

and

$$\mu_{m,n} = \inf_{\psi \in H^1_{loc}(0,\frac{\pi}{2})} \Big\{ \int_0^{\frac{\pi}{2}} |\psi'(\varphi)|^2 w_{m,n}(\varphi)\, d\varphi; \quad \int_0^{\frac{\pi}{2}} |\psi(\varphi)|^2 w_{m,n}(\varphi)\, d\varphi = 1, \int_0^{\frac{\pi}{2}} \psi(\varphi) w_{m,n}(\varphi)\, d\varphi = 0 \Big\}. \tag{104}$$

Let ψ_l be the unique minimizer in (103), and $\psi_{m,n}$ be the unique minimizer in (104). Let E be the formal Euler-Lagrange functional of (87).
Let u be the solution obtained in part 3 of Theorem 9.1. We divide the proof into two cases. We first show that u depends on θ in a nontrivial way provided

$$\frac{(p-1)\mu_l}{p-2} < \beta_0.$$

Then hen we show that u depends on θ in a non-trivial way provided

$$\frac{(p-1)\mu_{m,n}}{p-2} < \beta_0.$$

Case I. We proceed by way of contradiction. Let us assume that u is not a function of θ.
Set $v(r,\varphi,\theta) = u(r,\varphi)\psi_l(\theta)$. We just need to show that

$$\langle E''(u); v, v\rangle := \int_\Omega |\nabla v|^2\, dx - (p-1)\int_\Omega |a(|x|)u|^{p-2} v^2\, dx < 0. \tag{105}$$

Note first that $u = u(r,\varphi)$ satisfies the equation $-\Delta u = a(r)u^{p-1}$. Multiplying both sides of the equation by $u(r,\varphi)\psi_l^2(\theta)$ and integrating in spherical coordinates imply that

$$\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} (u_r^2 + \frac{u_\varphi^2}{r^2\sin^2(\theta)})\psi_l^2(\theta) d\mu(\varphi,\theta,r) = \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} a(r)u^p\psi_l^2(\theta) d\mu(\varphi,\theta,r). \tag{106}$$

It also follows from the definition of $\beta_0 = \beta_0(\Omega)$, the best constant in Hardy inequality (102) for the function $v = u(r,\varphi)\psi_l(\theta)$ that

$$\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(u_r^2\psi_l^2(\theta) + \frac{u_\varphi^2\psi_l^2(\theta)}{r^2\sin^2(\theta)} + \frac{u^2\psi_l'^2}{r^2}\right) d\mu(\varphi,\theta,r) \geq \beta_0 \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{u^2\psi_l^2(\theta)}{r^2} d\mu(\varphi,\theta,r). \tag{107}$$

It now follows that

$$\begin{aligned}
\langle E''(u); v, v\rangle &= \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(u_r^2\psi_l^2 + \frac{u_\varphi^2\psi_l^2}{r^2\sin^2(\theta)} + \frac{u^2\psi_l'^2}{r^2} - (p-1)a(r)u^p\psi_l^2\right) d\mu(\varphi,\theta,r)\\
&= \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{u^2\psi_l'^2}{r^2} d\mu(\varphi,\theta,r) - (p-2)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(u_r^2\psi_l^2 + \frac{u_\varphi^2\psi_l^2}{r^2\sin^2(\theta)}\right) d\mu(\varphi,\theta,r)\\
&= (p-1)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{u^2\psi_l'^2}{r^2} d\mu - (p-2)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(u_r^2\psi_l^2 + \frac{u_\varphi^2\psi_l^2}{r^2\sin^2(\theta)} + \frac{u^2\psi_l'^2}{r^2}\right) d\mu\\
&= (p-1)\mu_l\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \frac{u^2\psi_l^2}{r^2} d\mu - (p-2)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(u_r^2\psi_l^2 + \frac{u_\varphi^2\psi_l^2}{r^2\sin^2(\theta)} + \frac{u^2\psi_l'^2}{r^2}\right) d\mu\\
&\leq \left(\frac{(p-1)\mu_l}{\beta_0} - (p-2)\right) \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} \left(u_r^2\psi_l^2 + \frac{u_\varphi^2\psi_l^2}{r^2\sin^2(\theta)} + \frac{u^2\psi_l'^2}{r^2}\right) d\mu < 0
\end{aligned}$$

Case II. Similar to the previous case, we proceed by way of contradiction. Let us assume that u is not a function of φ. Set $v(r,\varphi,\theta) = u(r,\theta)\psi_{m,n}(\varphi)$. To conclude the proof we show that

$$\langle E^{''}(u); v, v\rangle := \int_\Omega |\nabla v|^2\,dx - (p-1)\int_\Omega |a(|x|)u|^{p-2}v^2\,dx < 0. \tag{108}$$

Note first that $u = u(r,\theta)$ satisfies the equation $-\Delta u = a(r)u^{p-1}$. Multiplying both sides of the equation by $u(r,\theta)\psi^2_{m,n}(\varphi)$ and integrating in spherical coordinates imply that

$$\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}(u_r^2 + \frac{u_\theta^2}{r^2})\psi^2_{m,n}(\varphi)d\mu(\varphi,\theta,r) = \int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}} a(r)u^p\psi^2_{m,n}(\varphi)d\mu(\varphi,\theta,r). \tag{109}$$

It also follows from the definition of $\beta_0 = \beta_0(\Omega)$, the best constant in Hardy inequality for the function $v = u(r,\theta)\psi_{m,n}(\varphi)$ that

$$\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(u_r^2\psi^2_{m,n}(\varphi) + \frac{u^2\psi'^2_{m,n}(\varphi)}{r^2\sin^2(\theta)} + \frac{u_\theta^2\psi^2_{m,n}}{r^2}\right)d\mu(\varphi,\theta,r) \geq \beta_0\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\frac{u^2\psi^2_{m,n}(\varphi)}{r^2}d\mu(\varphi,\theta,r). \tag{110}$$

As in the proof of case one can deduce that

$$\langle E^{''}(u); v, v\rangle \leq \left(\frac{(p-1)\mu_{m,n}}{\beta_0} - (p-2)\right)\int_{R_1}^{R_2}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\left(u_r^2\psi^2_{m,n}(\varphi) + \frac{u^2\psi'^2_{m,n}(\varphi)}{r^2\sin^2(\theta)} + \frac{u_\theta^2\psi^2_{m,n}}{r^2}\right)d\mu(\varphi,\theta,r) < 0$$

□

We recall the following result from [26] about the largeness of the best constant β_0 in the hardy inequality where the domain is an annulus.

Proposition 9.1. *[26]*

- *Let $R_1 = R$ and $R_2 = R+1$. Then β_0 is sufficiently large for large values of R.*
- *Let $R < \gamma(R)$ with $\frac{\gamma(R)}{R} \to 1$ as $R \to \infty$. With $\Omega_R = \{x \in \mathbb{R}^N : R < |x| < \gamma(R)\}$ then for large enough R the β_0 corresponding to Ω_R is sufficiently large.*

Corollary 9.5. *Let $p > 2$ and $N > 3$. Consider the problem (87) where $\Omega = \{x \in \mathbb{R}^N : R < |x| < R+1\}$ and $a \equiv 1$. For large values of R, there are at least*

$$\left\lfloor\frac{N}{2}\right\rfloor + \left\lfloor\frac{N-1}{2}\right\rfloor + \left\lfloor\frac{N-2}{2}\right\rfloor + ... + \left\lfloor\frac{N-k}{2}\right\rfloor, \qquad k = \left\lfloor\frac{N}{3}\right\rfloor,$$

positive non radial solutions. Here $\lfloor z\rfloor$ stands for the floor of $z \in \mathbb{R}$.

Proof. Here we are going to use the $K_{-,\frac{\pi}{2}}$ symmetry in Theorem 9.4 and therefore m and n can be different. The cardinality of the set

$$D_2 = \{(m,n) \in \mathbb{N}\times\mathbb{N};\ m+n = \mathbb{N},\ 1 \leq m \leq n\}$$

is $\left\lfloor\frac{N}{2}\right\rfloor$, and for each $(m,n) \in D_2$ there exists a no-radial solution u which is invariant in $O(m)\times O(n)$ when R is large enough as we have shown in [26]. Also, the cardinality of the set

$$D_3 = \{(m,n,l) \in \mathbb{N}\times\mathbb{N}\times\mathbb{N};\ m+n+l = \mathbb{N},\ 1 \leq m \leq n \leq l\}$$

is

$$\Big\lfloor \frac{N-1}{2} \Big\rfloor + \Big\lfloor \frac{N-2}{2} \Big\rfloor + ... + \Big\lfloor \frac{N-k}{2} \Big\rfloor, \qquad k = \Big\lfloor \frac{N}{3} \Big\rfloor.$$

By Theorem 9.4, for each $(m, n, l) \in D_3$ there exists a solution u which is invariant in $O(m) \times O(n) \times O(l)$ and it is non invariant in $O(j) \times O(N-j)$ for any $j \in \{1, ..., N-1\}$. This completes the proof. □

9.1.2 The case of symmetry in φ

In this section we examine the case where the domain and a have symmetry in φ. In terms of the coordinates (s, t, τ) we are examining the case where we have cylindrical symmetry around the τ axis. We suppose $m = n$ and Ω satisfies assumption 2 from Definition 9.2, ie. suppose $g^i = g^i(\varphi, \theta)$ is smooth and positive on $[0, \frac{\pi}{2}] \times [0, \pi/2]$ and for each fixed $\theta \in (0, \pi/2)$ and $i = 1, 2$ we have $\varphi \mapsto g^i(\varphi, \theta)$ is constant on $(0, \pi/2)$.

We further assume that $a = a(r, \theta)$. Then looking at (87) (written in terms of (r, φ, θ)) one sees that it is reasonable to look for solutions of (87) which don't depend on φ and in fact one can use the same imbedding to obain a solution for the given range of paramters that doesn't depend on φ. Our next theorem gives sufficient conditions under which the ground state solution depends on φ in a nontrivial way.

Theorem 9.5. *Suppose Ω satisfies the above hypothesis and $a_\varphi = 0$.*

1. *Suppose p satisfies hypothesis from Theorem 9.1 part 2 and u is K_+ ground state solution promised by Theorem 9.1 part 2. If β_0 is large enough then u is a function that depends on φ in a nontrivial way.*

2. *Suppose p satisfies hypothesis from Theorem 9.1 part 1 and u is K_- ground state solution promised by Theorem 9.1 part 1. If β_0 is large enough then u is a function that depends on φ in a nontrivial way.*

Proof. The proof follows the same strategy as the proof of Theorem 9.4. □

Remark 9.6. *One can examine multiplicity type results for these domains also, we leave this to the interested reader.*

References

[1] A. Aghajani, C. Cowan and A. Moameni, *The Gelfand problem on annular domains of double revolution with monotonicity*, accepted Proc. AMS, (2022).

[2] A. Aghajani, S. Ai and C. Cowan, *A singular potential problem*, (2021) preprint.

[3] C. O. Alves, A. Moameni, *Super-critical Neumann problems on unbounded domains.* Nonlinearity 33 (2020), no. 9, 4568-4589.

[4] M. Badiale, E. Serra, *Multiplicity results for the supercritical Hénon equation Advanced Nonlinear Studies*, 4/4 (2004), 453-467.

[5] Bartsch, Thomas; Willem, Michel. *Infinitely many nonradial solutions of a Euclidean scalar field equation.* J. Funct. Anal. 117 (1993), no. 2, 447-460.

[6] A. Bahri and J.M. Coron, *On a nonlinear elliptic equation involving the critical sobolev exponent: The effect of the topology of the domain*, Comm. Pure Appl. Math., 41: 253-294, (1988).

[7] Bartsch, T., Clapp, M., Grossi, F. Pacella, *Asymptotically radial solutions in expanding annular domains*, Math. Ann. 352, 485–515 (2012).

[8] V. Barutello, S. Secchi and E. Serra, *A note on the radial solutions for the supercritical Hénon equation*, J. Math. Anal. Appl. 341(1) (2008), 720-728.

[9] D. Bonheure, M. Grossi, B. Noris and S. Terracini, *Multi-layer radial solutions for a supercritical Neumann problem*, J. Differential Equations, 261(1):455-504, 2016.

[10] D. Bonheure, B. Noris and T. Weth, *Increasing radial solutions for Neumann problems without growth restrictions*, Ann. Inst. H. Poincaré Anal. Non Linéaire 29 (2012), 4, 573-588.

[11] D. Bonheure and E. Serra, *Multiple positive radial solutions on annuli for nonlinear Neumann problems with large growth*, NoDEA 18 (2011), 2, 217-235.

[12] A. Boscaggin, F. Colasuonno, B. Noris and Tobias Weth, *A supercritical elliptic equation in the annulus*, arXiv:2102.07141, (2021).

[13] J. Byeon, *Existence of many nonequivalent nonradial positive solutions of semilinear elliptic equations on three-dimensional annuli*, J. Differ. Equ. 136, 136–165 (1997)

[14] F. Catrina, Z.Q. Wang, *Nonlinear elliptic equations on expanding symmetric domains.* J. Differ. Equ. 156, 153–181 (1999)

[15] M. Clapp and A. Pistoia, *Symmetries, Hopf fibrations and supercritical elliptic problems*, Contemp. Math 656, 1-12, (2016).

[16] M. Clapp, *A global compactness result for elliptic problems with critical nonlinearity on symmetric domains*, Nonlinear Equations: Methods, Models and Applications, 117-126, (2003).

[17] M. Clapp and J. Faya, *Multiple solutions to the Bahri-Coron problem in some domains with nontrivial topology*, Proc. Amer. Math. Soc. 141 (2013), 4339-4344.

[18] M. Clapp, J. Faya, and A. Pistoia, *Nonexistence and multiplicity of solutions to elliptic problems with supercritical exponents*, Calc. Var. Partial Differential Equations 48 (2013), 611-623.

[19] M. Clapp, J. Faya, and A. Pistoia, *Positive solutions to a supercritical elliptic problem which concentrate along a thin spherical hole*, J. Anal. Math.,126, pages341-357 (2015).

[20] M. Clapp and F. Pacella, *Multiple solutions to the pure critical exponent problem in domains with a hole of arbitrary size*, Math. Z. 259 (2008), 575-589.

[21] F. Colasuonno and B. Noris. *A p-Laplacian supercritical Neumann problem.* Discrete and Continuous Dynamical Systems, 37 (2017), (6): 3025-3057.

[22] J.M. Coron, *Topologie et cas limite des injections de Sobolev.* C.R. Acad. Sc. Paris, 299, Series I, 209–212.(1984).

[23] C. Cowan, *Optimal Hardy inequalities for general elliptic operators with improvements*, Commun. Pure Appl. Anal. 9 (2010), no. 1, 109-140.

[24] C. Cowan, *Supercritical elliptic problems on a perturbation of the ball.* J. Diff. Eq. 256 (2014), 3, 1250-1263.

[25] C. Cowan and A. Moameni, *A new variational principle, convexity, and supercritical Neumann problems*, Transactions of the American Mathematical Society 371 (2019), (9), 5993-6023.

[26] C. Cowan and A. Moameni, *Supercritical elliptic problems on nonradial domains via a nonsmooth variational approach*, (2021) preprint.

[27] C. Cowan, A. Moameni and L. Salimi, *Supercritical Neumann problems via a new variational principle*, Electron. J. Differential Equations 2017 (213), 1-19.

[28] M. del Pino, M. Musso and A. Pistoia, *Super-critical boundary bubbling in a semilinear Neumann problem*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis Volume 22, Issue 1 (2005), 45-82.

[29] M. del Pino, J. Dolbeault and M. Musso, *A phase plane analysis of the multi-bubbling phenomenon in some slightly supercritical equations*, Monatsh. Math. 142 no. 1-2, 57-79 (2004).

[30] M. del Pino, J. Dolbeault and M. Musso, *Bubble-tower radial solutions in the slightly supercritical Brezis-Nirenberg problem*, Journal of Differential Equations 193 (2003), no. 2, 280-306.

[31] M. del Pino, P. Felmer, M. Musso, Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries Bull. London Math. Society 35 (2003), no. 4, 513-521

[32] M. del Pino, P. Felmer and M. Musso, *Two-bubble solutions in the super-critical Bahri-Coron's problem.* Calc. Var. Partial Differential Equations 16 (2003), (2):113–145.

[33] M. del Pino and J. Wei, *Supercritical elliptic problems in domains with small holes,* Ann. Non linearie, Annoles de l'Institut H. Poincare, 24 (2007), no.4, 507-520.

[34] P. Esposito, A. Pistoia, J. Wei, *Concentrating solutions for the Hénon equation in R^2,* (2006) Journal d'Analyse Mathematique, 100, pp. 249-80.

[35] Ekeland, I. and Temam, R. *Convex Analysis and Variational Problems.* Classics in Applied Mathematics, 28. SIAM, (1999).

[36] F. Gladiali and M. Grossi, *Supercritical elliptic problem with nonautonomous nonlinearities,* J. Diff. Eq. 253 (2012), 2616-2645.

[37] F. Gladiali, M. Grossi, F. Pacella, P. N. Srikanth, *Bifurcation and symmetry breaking for a class of semilinear elliptic equations in an annulus.* Calc. (2011) Var. 40, 295–317.

[38] M. Grossi, *A class of solutions for the Neumann problem* $-\Delta + \lambda u = u^{\frac{N+2}{N-2}}$ Duke Math. J., 79 (2) (1995), 309-334.

[39] M. Grossi and B. Noris, *Positive constrained minimizers for supercritical problems in the ball*, Proc. AMS, 140 (2012), 6, 2141-2154.

[40] C. Gui, *Multi-peak solutions for a semilinear Neumann problem*, Duke Math. J., 84 (1996), 739-769.

[41] C. Gui and C.-S. Lin, *Estimates for boundary-bubbling solutions to an elliptic Neumann problem* J. Reine Angew. Math., 546 (2002), 201-235.

[42] N. Ghoussoub and C. Gui, *Multi-peak solutions for a semilinear Neumann problem involving the critical Sobolev exponent* Math. Z., 229 (3) (1998), 443-474.

[43] C. Gui and J. Wei, *Multiple interior peak solutions for some singularly perturbed Neumann problems*, J. Differential Equations, 158 (1) (1999), 1-27.

[44] J. Hao, X. Chen and Y. Zhang, *Infinitely many spike solutions for the Hénon equation with critical growth*, Journal of Differential Equations, 259/9 (2015), 4924-4946.

[45] Y.Y. Li, *Existence of many positive solutions of semilinear elliptic equations on annulus.* J. Differ. Equ. 83, 348–367 (1990)

[46] Lions, Pierre-Louis, *Symmetry and compactness in Sobolev spaces.* J. Functional Analysis 49 (1982), no. 3, 315-334.

[47] Y. Lu, T. Chen, and R. Ma, *On the Bonheure-Noris-Weth conjecture in the case of linearly bounded nonlinearities,* Discrete Contin. Dyn. Syst. Ser. B, 21 (2016), (8):2649-2662.

[48] N. Kouhestani, A. Moameni, *Multiplicity results for elliptic problems with super-critical concave and convex nonlinearties.* Calc. Var. (2018) 57:54.

[49] J. McGough, J. Mortensen, *Pohozaev obstructions on non-starlike domains.* Calc. Var. Partial Differential Equations 18 (2003), no. 2, 189–205.

[50] A. Moameni, *New variational principles of symmetric boundary value problems*, Journal of Convex Analysis, 24 (2017) 365–381.

[51] A. Moameni, *Critical point theory on convex subsets with applications in differential equations and analysis.* J. Math. Pures Appl. (9) 141 (2020), 266-315.

[52] W.M. Ni, *A Nonlinear Dirichlet problem on the unit ball and its applications,* Indiana Univ. Math. Jour. 31 (1982), 801-807.

[53] D. Passaseo, *Nonexistence results for elliptic problems with supercritical nonlinearity in non-trivial domains.* J. Funct. Anal. 114(1):97–105.(1993).

[54] D. Passaseo. *Existence and multiplicity of positive solutions for elliptic equations with supercritical nonlinearity in contractible domains.* Rend. Accad. Naz. Sci. XL Mem. Mat. (5), 16:77–98, 1992.

[55] S.J. Peng, *Multiple boundary concentrating solutions to dirichlet problem of Hénon equation*, Acta Mathematicae Applicatae Sinica, 22/1 (2006),

[56] A. Pistoia, E. Serra, *Multi-peak solutions for the Hénon equation with slightly subcritical growth*, Mathematische Zeitschrift, 256/1 (2007).

[57] S. Pohozaev, *Eigenfunctions of the equation* $-\Delta u+\lambda f(u)=0$*,* Soviet Math. Doklady 6 (1965), 1408-1411.

[58] O. Rey and J. Wei, *Blowing up solutions for an elliptic Neumann problem with sub- or super-critical nonlinearity Part I:* $N=3$, Journal of Functional Analysis Volume 212, Issue 2, 15 (2004), 472-499.

[59] R. Schaaf, *Uniqueness for semilinear elliptic problems: supercritical growth and domain geometry*, Adv. Differential Equations, 5:10–12 (2000), pp. 1201–1220.

[60] K. Schmitt, *Positive solutions of semilinear elliptic boundary value problems*, Topological methods in differential equations and inclusions (Montreal, PQ, 1994), NATO Adv. Sci. Inst. Ser. C Math. Phys. Sci., vol. 472, Kluwer Academic Publishers, Dordrecht, 1995, pp. 447-500.

[61] E. Serra and P. Tilli, *Monotonicity constraints and supercritical Neumann problems*, Annales de l'Institut Henri Poincare (C) Non Linear Analysis, 28 (2011), 63-74.

[62] Smets, D., Willem, M. and Su, J. (2002). *Non-radial ground states for the Hénon equation. Communications in Contemporary Mathematics*, 4(03), 467-480.

[63] A. Szulkin, *Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems.* Ann. Inst. H. Poincaré Anal. Non Linéaire 3 (1986), no. 2, 77-109.

[64] J. F. Toland, *A duality principle for non-convex optimisation and the calculus of variations.* Arch. Rational Mech. Anal., vol. 71 (1979).

[65] J. Wei, *On the boundary spike layer solutions to a singularly perturbed Neumann problem*, J. Differential Equations, 134 (1) (1997), 104-133.