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ON STABLE ENTIRE SOLUTIONS OF SEMI-LINEAR ELLIPTIC EQUATIONS WITH WEIGHTS

CRAIG COWAN AND MOSTAFA FAZLY

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ABSTRACT. We are interested in the existence versus non-existence of non-trivial stable sub- and super-solutions of

$$-\operatorname{div}(\omega_1 \nabla u) = \omega_2 f(u) \qquad \text{in } \mathbb{R}^N, \tag{0.1}$$

with positive smooth weights $\omega_1(x), \omega_2(x)$. We consider the cases $f(u) = e^u, u^p$ where $p > 1$ and $-u^{-p}$ where $p > 0$. We obtain various non-existence results which depend on the dimension N and also on p and the behaviour of ω_1, ω_2 near infinity. Also the monotonicity of ω_1 is involved in some results. Our methods here are the methods developed by Farina. We examine a specific class of weights $\omega_1(x) = (|x|^2+1)^{\frac{\alpha}{2}}$ and $\omega_2(x) = (|x|^2+1)^{\frac{\beta}{2}} g(x)$, where $g(x)$ is a positive function with a finite limit at ∞. For this class of weights, non-existence results are optimal. To show the optimality we use various generalized Hardy inequalities.

1. Introduction and main results

In this note we are interested in the existence versus non-existence of stable sub- and super-solutions of equations of the form

$$-\operatorname{div}(\omega_1(x) \nabla u) = \omega_2(x) f(u) \qquad \text{in } \mathbb{R}^N, \tag{1.1}$$

where $f(u)$ is one of the following non-linearities: e^u, u^p where $p > 1$ and $-u^{-p}$ where $p > 0$. We assume that $\omega_1(x)$ and $\omega_2(x)$, which we call *weights*, are smooth positive functions (we allow ω_2 to be zero at say a point) and which satisfy various growth conditions at ∞. Recall that we say that a solution u of $-\Delta u = f(u)$ in $\mathbb{R}^N$ is stable provided that

$$\int f'(u)\psi^2 \le \int |\nabla \psi|^2, \qquad \forall \psi \in C_c^2,$$

where C_c^2 is the set of C^2 functions defined on $\mathbb{R}^N$ with compact support. Note that the stability of u is just saying that the second variation at u of the energy

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associated with the equation is non-negative. In our setting this becomes: We say that a C^2 sub/super-solution u of (1.1) is *stable* provided that

$$\int \omega_2 f'(u)\psi^2 \leq \int \omega_1 |\nabla\psi|^2 \qquad \forall \psi \in C_c^2. \tag{1.2}$$

One should note that (1.1) can be rewritten as

$$-\Delta u + \nabla\gamma(x)\cdot\nabla u = \omega_2/\omega_1 \; f(u) \qquad \text{in } \mathbb{R}^N,$$

where $\gamma = -\log(\omega_1)$, and on occasion we shall take this point of view.

Remark 1. Note that if ω_1 has enough integrability, then it is immediate that if u is a stable solution of (1.1) we have $\int \omega_2 f'(u) = 0$ (provided f is increasing). To see this let $0 \leq \psi \leq 1$ be supported in a ball of radius $2R$ centered at the origin (B_{2R}) with $\psi = 1$ on B_R and such that $|\nabla\psi| \leq \frac{C}{R}$, where $C > 0$ is independent of R. Putting this ψ into (1.2) one obtains

$$\int_{B_R} \omega_2 f'(u) \leq \frac{C}{R^2}\int_{R<|x|<2R} \omega_1,$$

and so if the right-hand side goes to zero as $R \to \infty$ we have the desired result.

The existence versus non-existence of stable solutions of $-\Delta u = f(u)$ in $\mathbb{R}^N$ or $-\Delta u = g(x)f(u)$ in $\mathbb{R}^N$ is now quite well understood; see [3, 4, 7, 8, 9, 10, 11, 1, 5, 6]. We remark that some of these results are examining the case where Δ is replaced with Δ_p (the p-Laplacian), and also in many cases the authors are interested in finite Morse index solutions or solutions which are stable outside a compact set. Much of the interest in these Liouville type theorems stems from the fact that the non-existence of a stable solution is related to the existence of a priori estimates for stable solutions of a related equation on a bounded domain.

In [12] equations similar to $-\Delta u = |x|^\alpha u^p$ were examined on the unit ball in $\mathbb{R}^N$ with zero Dirichlet boundary conditions. There it was shown that for $\alpha > 0$ one can obtain positive solutions for p supercritical with respect to Sobolev embedding and so one can observe that the term $|x|^\alpha$ is restoring some compactness. A similar feature happens for equations of the form

$$-\Delta u = |x|^\alpha f(u) \qquad \text{in } \mathbb{R}^N;$$

the value of α can vastly alter the existence versus non-existence of a stable solution; see [5, 1, 6, 8, 7].

We now come to our main results and for this we need to define a few quantities:

$$\begin{aligned}
I_G &:= R^{-4t-2}\int_{R<|x|<2R}\frac{\omega_1^{2t+1}}{\omega_2^{2t}}dx,\\
J_G &:= R^{-2t-1}\int_{R<|x|<2R}\frac{|\nabla\omega_1|^{2t+1}}{\omega_2^{2t}}dx,\\
I_L &:= R^{\frac{-2(2t+p-1)}{p-1}}\int_{R<|x|<2R}\left(\frac{w_1^{p+2t-1}}{w_2^{2t}}\right)^{\frac{1}{p-1}}dx,\\
J_L &:= R^{-\frac{p+2t-1}{p-1}}\int_{R<|x|<2R}\left(\frac{|\nabla w_1|^{p+2t-1}}{w_2^{2t}}\right)^{\frac{1}{p-1}}dx,\\
I_M &:= R^{-2\frac{p+2t+1}{p+1}}\int_{R<|x|<2R}\left(\frac{w_1^{p+2t+1}}{w_2^{2t}}\right)^{\frac{1}{p+1}}dx,\\
J_M &:= R^{-\frac{p+2t+1}{p+1}}\int_{R<|x|<2R}\left(\frac{|\nabla w_1|^{p+2t+1}}{w_2^{2t}}\right)^{\frac{1}{p+1}}dx.
\end{aligned}$$

The three equations we examine are

$$\begin{aligned}
-\operatorname{div}(\omega_1\nabla u) &= \omega_2 e^u \qquad \text{in } \mathbb{R}^N \quad (G),\\
-\operatorname{div}(\omega_1\nabla u) &= \omega_2 u^p \qquad \text{in } \mathbb{R}^N \quad (L),\\
-\operatorname{div}(\omega_1\nabla u) &= -\omega_2 u^{-p} \qquad \text{in } \mathbb{R}^N \quad (M),
\end{aligned}$$

where we restrict (L) to the case $p>1$ and (M) to $p>0$. By a solution we always mean a C^2 solution. We now come to our main results in terms of abstract ω_1 and ω_2. We remark that our approach to non-existence of stable solutions is the approach due to Farina; see [9, 10, 4].

Theorem 1.1.

(1) *There is no stable sub-solution of* (G) *if* $I_G, J_G \to 0$ *as* $R\to\infty$ *for some* $0<t<2$.

(2) *There is no positive stable sub-solution (super-solution) of* (L) *if* $I_L, J_L \to 0$ *as* $R\to\infty$ *for some* $p-\sqrt{p(p-1)}<t<p+\sqrt{p(p-1)}$ $(0<t<\frac{1}{2})$.

(3) *There is no positive stable super-solution of (M) if* $I_M, J_M\to 0$ *as* $R\to\infty$ *for some* $0<t<p+\sqrt{p(p+1)}$.

If we assume that ω_1 has some monotonicity we can do better. We will assume that the monotonicity conditions are satisfied for large x but really all one needs is for it to be satisfied on a suitable sequence of annuli.

Theorem 1.2.

(1) *There is no stable sub-solution of* (G) *with* $\nabla\omega_1(x)\cdot x\le 0$ *for large* x *if* $I_G\to 0$ *as* $R\to\infty$ *for some* $0<t<2$.

(2) *There is no positive stable sub-solution of* (L) *provided* $I_L\to 0$ *as* $R\to\infty$ *for either:*
  - *some* $1\le t<p+\sqrt{p(p-1)}$ *and* $\nabla\omega_1(x)\cdot x\le 0$ *for large* x *or*
  - *some* $p-\sqrt{p(p-1)}<t\le 1$ *and* $\nabla\omega_1(x)\cdot x\ge 0$ *for large* x.

  There is no positive super-solution of (L) *provided* $I_L\to 0$ *as* $R\to\infty$ *for some* $0<t<\frac{1}{2}$ *and* $\nabla\omega_1(x)\cdot x\le 0$ *for large* x.

(3) *There is no positive stable super-solution of* (M) *provided* $I_M \to 0$ *as* $R \to \infty$ *for some* $0 < t < p + \sqrt{p(p+1)}$.

Corollary 1. *Suppose* $\omega_1 \le C\omega_2$ *for large* x, $\omega_2 \in L^\infty$, $\nabla \omega_1(x) \cdot x \le 0$ *for large* x.

(1) *There is no stable sub-solution of* (G) *if* $N \le 9$.
(2) *There is no positive stable sub-solution of* (L) *if*

$$N < 2 + \frac{4}{p-1}\left(p + \sqrt{p(p-1)}\right).$$

(3) *There is no positive stable super-solution of* (M) *if*

$$N < 2 + \frac{4}{p+1}\left(p + \sqrt{p(p+1)}\right).$$

If one takes $\omega_1 = \omega_2 = 1$ in the above corollary, the results obtained for (G) and (L), and for some values of p in (M), are optimal; see [9, 10, 8].

We now drop all monotonicity conditions on ω_1.

Corollary 2. *Suppose* $\omega_1 \le C\omega_2$ *for large* x, $\omega_2 \in L^\infty$, $|\nabla \omega_1| \le C\omega_2$ *for large* x.

(1) *There is no stable sub-solution of* (G) *if* $N \le 4$.
(2) *There is no positive stable sub-solution of* (L) *if*

$$N < 1 + \frac{2}{p-1}\left(p + \sqrt{p(p-1)}\right).$$

(3) *There is no positive super-solution of* (M) *if*

$$N < 1 + \frac{2}{p+1}\left(p + \sqrt{p(p+1)}\right).$$

Some of the conditions on ω_i in Corollary 2 seem somewhat artificial. If we shift over to the advection equation (and we take $\omega_1 = \omega_2$ for simplicity)

$$-\Delta u + \nabla \gamma \cdot \nabla u = f(u),$$

the conditions on γ become: γ is bounded from below and has a bounded gradient.

In what follows we examine the case where $\omega_1(x) = (|x|^2+1)^{\frac{\alpha}{2}}$ and $\omega_2(x) = g(x)(|x|^2+1)^{\frac{\beta}{2}}$, where $g(x)$ is positive, except at say a point, and smooth, and where $\lim_{|x|\to\infty} g(x) = C \in (0,\infty)$. For this class of weights we can essentially obtain optimal results.

Theorem 1.3. *Take* ω_1 *and* ω_2 *as above.*

(1) *If* $N + \alpha - 2 < 0$, *then there is no stable sub-solution for* (G), (L) *(here we require it to be positive), and in the case of* (M) *there is no positive stable super-solution. This case is the trivial case; see Remark* 1.

Assumption. *For the remaining cases we assume that* $N + \alpha - 2 > 0$.

(2) *If* $N + \alpha - 2 < 4(\beta - \alpha + 2)$, *then there is no stable sub-solution for* (G).
(3) *If* $N + \alpha - 2 < \frac{2(\beta-\alpha+2)}{p-1}\left(p + \sqrt{p(p-1)}\right)$, *then there is no positive stable sub-solution of* (L).
(4) *If* $N + \alpha - 2 < \frac{2(\beta-\alpha+2)}{p+1}\left(p + \sqrt{p(p+1)}\right)$, *then there is no positive stable super-solution of* (M).

(5) *Furthermore,* (2), (3), (4) *are optimal in the sense that if* $N+\alpha-2>0$ *and the remaining inequality is not satisfied (and in addition we assume we do not have equality in the inequality), then we can find a suitable function* $g(x)$ *which satisfies the above properties and a stable sub-/super-solution* u *for the appropriate equation.*

Remark 2. Many of the above results can be extended to the case of equality in either $N+\alpha-2\geq 0$ or the other inequality which depends on the equation we are examining. We omit the details because one cannot prove the results in a unified way.

In showing that an explicit solution is stable we will need the weighted Hardy inequality given in [2].

Lemma 1. *Suppose* $E>0$ *is a smooth function. Then one has*

$$(\tau-\frac{1}{2})^2\int E^{2\tau-2}|\nabla E|^2\phi^2+(\frac{1}{2}-\tau)\int(-\Delta E)E^{2\tau-1}\phi^2\leq\int E^{2\tau}|\nabla\phi|^2,$$

for all $\phi\in C_c^\infty(\mathbb{R}^N)$ *and* $\tau\in\mathbb{R}$.

By picking an appropriate function E this gives

Corollary 3. *For all* $\phi\in C_c^\infty$ *and* $t,\alpha\in\mathbb{R}$, *we have*

$$\begin{aligned}\int(1+|x|^2)^{\frac{\alpha}{2}}|\nabla\phi|^2 \quad\geq\quad & (t+\frac{\alpha}{2})^2\int|x|^2(1+|x|^2)^{-2+\frac{\alpha}{2}}\phi^2\\ & +(t+\frac{\alpha}{2})\int(N-2(t+1)\frac{|x|^2}{1+|x|^2})(1+|x|^2)^{-1+\frac{\alpha}{2}}\phi^2.\end{aligned}$$

2. Proof of main results

Proof of Theorem 1.1. (1) Suppose u is a stable sub-solution of (G) with $I_G,J_G\to 0$ as $R\to\infty$ and let $0\leq\phi\leq 1$ denote a smooth compactly supported function. Put $\psi:=e^{tu}\phi$ into (1.2), where $0<t<2$, to arrive at

$$\begin{aligned}\int\omega_2e^{(2t+1)u}\phi^2 \quad\leq\quad & t^2\int\omega_1e^{2tu}|\nabla u|^2\phi^2\\ & +\int\omega_1e^{2tu}|\nabla\phi|^2+2t\int\omega_1e^{2tu}\phi\nabla u\cdot\nabla\phi.\end{aligned}$$

Now multiply (G) by $e^{2tu}\phi^2$ and integrate by parts to arrive at

$$2t\int\omega_1e^{2tu}|\nabla u|^2\phi^2\leq\int\omega_2e^{(2t+1)u}\phi^2-2\int\omega_1e^{2tu}\phi\nabla u\cdot\nabla\phi,$$

and now if one equates like terms, one arrives at

$$\begin{aligned}\frac{(2-t)}{2}\int\omega_2e^{(2t+1)u}\phi^2 \quad\leq\quad & \int\omega_1e^{2tu}\left(|\nabla\phi|^2-\frac{\Delta\phi}{2}\right)dx\\ & -\frac{1}{2}\int e^{2tu}\phi\nabla\omega_1\cdot\nabla\phi.\end{aligned}\tag{2.1}$$

Now substitute ϕ^m into this inequality for ϕ where m is a large integer to obtain

$$\begin{aligned}\frac{(2-t)}{2}\int\omega_2e^{(2t+1)u}\phi^{2m} \quad\leq\quad & C_m\int\omega_1e^{2tu}\phi^{2m-2}\left(|\nabla\phi|^2+\phi|\Delta\phi|\right)dx\\ & -D_m\int e^{2tu}\phi^{2m-1}\nabla\omega_1\cdot\nabla\phi,\end{aligned}\tag{2.2}$$

where C_m and D_m are positive constants just depending on m. We now estimate the terms on the right, but we mention that when one assumes the appropriate monotonicity on ω_1, it is the last integral on the right which one is able to drop:

$$\begin{aligned}\int \omega_1 e^{2tu}\phi^{2m-2}|\nabla\phi|^2 &= \int \omega_2^{\frac{2t}{2t+1}} e^{2tu}\phi^{2m-2}\frac{\omega_1}{\omega_2^{\frac{2t}{2t+1}}}|\nabla\phi|^2 \\ &\le \left(\int \omega_2 e^{(2t+1)u}\phi^{(2m-2)\frac{(2t+1)}{2t}}dx\right)^{\frac{2t}{2t+1}} \\ &\quad\times\left(\int \frac{\omega_1^{2t+1}}{\omega_2^{2t}}|\nabla\phi|^{2(2t+1)}\right)^{\frac{1}{2t+1}}.\end{aligned}$$

Now, for fixed $0 < t < 2$ we can take m large enough so that $(2m-2)\frac{(2t+1)}{2t} \ge 2m$, and since $0 \le \phi \le 1$ this allows us to replace the power on ϕ in the first term on the right with $2m$, and hence we obtain

(2.3)
$$\int \omega_1 e^{2tu}\phi^{2m-2}|\nabla\phi|^2 \le \left(\int \omega_2 e^{(2t+1)u}\phi^{2m}dx\right)^{\frac{2t}{2t+1}}\left(\int \frac{\omega_1^{2t+1}}{\omega_2^{2t}}|\nabla\phi|^{2(2t+1)}\right)^{\frac{1}{2t+1}}.$$

We now take the test functions ϕ to be such that $0 \le \phi \le 1$ with ϕ supported in the ball B_{2R} with $\phi = 1$ on B_R and $|\nabla\phi| \le \frac{C}{R}$, where $C > 0$ is independent of R. Using this choice of ϕ we obtain

$$\int \omega_1 e^{2tu}\phi^{2m-2}|\nabla\phi|^2 \le \left(\int \omega_2 e^{(2t+1)u}\phi^{2m}\right)^{\frac{2t}{2t+1}} I_G^{\frac{1}{2t+1}}. \tag{2.4}$$

One similarly shows that

$$\int \omega_1 e^{2tu}\phi^{2m-1}|\Delta\phi| \le \left(\int \omega_2 e^{(2t+1)u}\phi^{2m}\right)^{\frac{2t}{2t+1}} I_G^{\frac{1}{2t+1}}.$$

So, combining the results we obtain

$$\begin{aligned}\frac{(2-t)}{2}\int \omega_2 e^{(2t+1)u}\phi^{2m} &\le C_m\left(\int \omega_2 e^{(2t+1)u}\phi^{2m}dx\right)^{\frac{2t}{2t+1}} I_G^{\frac{1}{2t+1}} \\ &\quad -D_m\int e^{2tu}\phi^{2m-1}\nabla\omega_1\cdot\nabla\phi.\end{aligned} \tag{2.5}$$

We now estimate this last term. A similar argument using Hölder's inequality shows that

$$\int e^{2tu}\phi^{2m-1}|\nabla\omega_1||\nabla\phi| \le \left(\int \omega_2\phi^{2m}e^{(2t+1)u}dx\right)^{\frac{2t}{2t+1}} J_G^{\frac{1}{2t+1}}.$$

Combining the results gives that

$$(2-t)\left(\int \omega_2 e^{(2t+1)u}\phi^{2m}dx\right)^{\frac{1}{2t+1}} \le I_G^{\frac{1}{2t+1}} + J_G^{\frac{1}{2t+1}}, \tag{2.6}$$

and now we send $R \to \infty$ and use the fact that $I_G, J_G \to 0$ as $R \to \infty$ to see that

$$\int \omega_2 e^{(2t+1)u} = 0,$$

which is clearly a contradiction. Hence there is no stable sub-solution of (G).

(2) Suppose that $u > 0$ is a stable sub-solution (super-solution) of (L). Then a similar calculation as in (1) shows that for $p - \sqrt{p(p-1)} < t < p + \sqrt{p(p-1)}$ $(0 < t < \frac{1}{2})$ one has

$$(p - \frac{t^2}{2t-1}) \int \omega_2 u^{2t+p-1} \phi^{2m} \leq D_m \int \omega_1 u^{2t} \phi^{2(m-1)} (|\nabla \phi|^2 + \phi |\Delta \phi|) + C_m \frac{(1-t)}{2(2t-1)} \int u^{2t} \phi^{2m-1} \nabla \omega_1 \cdot \nabla \phi.$$

One now applies Hölder's argument as in (1), but the terms I_L and J_L will appear on the right-hand side of the resulting equation. This shift from a sub-solution to a super-solution depending on whether $t > \frac{1}{2}$ or $t < \frac{1}{2}$ is a result from the sign change of $2t - 1$ at $t = \frac{1}{2}$. We leave the details for the reader.

(3) This case is also similar to (1) and (2). □

Proof of Theorem 1.2. (1) Again we suppose that there is a stable sub-solution u of (G). Our starting point is (2.2), and we wish to be able to drop the term

$$-D_m \int e^{2tu} \phi^{2m-1} \nabla \omega_1 \cdot \nabla \phi$$

from (2.2). We can choose ϕ as in the proof of Theorem 1.1 but also such that $\nabla \phi(x) = -C(x)x$, where $C(x) \geq 0$. So if we assume that $\nabla \omega_1 \cdot x \leq 0$ for large x, then we see that this last term is non-positive and hence we can drop the term. Then the proof is as before, but now we only require that $\lim_{R \to \infty} I_G = 0$.

(2) Suppose that $u > 0$ is a stable sub-solution of (L) and so (2.7) holds for all $p - \sqrt{p(p-1)} < t < p + \sqrt{p(p-1)}$. Now we wish to use monotonicity to drop the term from (2.7) involving the term $\nabla \omega_1 \cdot \nabla \phi$. ϕ is chosen similarly as in (1), but here one notes that the coefficient for this term changes sign at $t = 1$, and hence by restriction of t to the appropriate side of (1) (along with the above condition on t and ω_1) we can drop the last term depending on which monotonicity we have and hence obtain a contraction we only require that $\lim_{R \to \infty} I_L = 0$. The result for the non-existence of a stable super-solution is similar, but here one uses the restriction $0 < t < \frac{1}{2}$.

(3) The proof here is similar to (1) and (2), and we omit the details. □

Proof of Corollary 1. We suppose that $\omega_1 \leq C \omega_2$ for large x, $\omega_2 \in L^\infty$, $\nabla \omega_1(x) \cdot x \leq 0$ for large x.

(1). Since $\nabla \omega_1 \cdot x \leq 0$ for large x we can apply Theorem 1.2 to show the non-existence of a stable solution to (G). Note that with the above assumptions on ω_i we have that

$$I_G \leq \frac{C R^N}{R^{4t+2}}.$$

For $N \leq 9$ we can take $0 < t < 2$ but close enough to 2 so that the right-hand side goes to zero as $R \to \infty$.

Both (2) and (3) also follow directly from applying Theorem 1.2. Note that one can say more about (2) by taking the multiple cases as listed in Theorem 1.2, but we have chosen to leave this to the reader. □

Proof of Corollary 2. Since we have no monotonicity conditions now we will need both I and J to go to zero to show the non-existence of a stable solution. Again the results are obtained immediately by applying Theorem 1.1, and we prefer to omit the details. □

Proof of Theorem 1.3. (1) If $N+\alpha-2<0$, then using Remark 1 one easily sees there is no stable sub-solution of (G) and (L) (positive for (L)) or a positive stable super-solution of (M). So we now assume that $N+\alpha-2>0$. Note that the monotonicity of ω_1 changes when α changes sign, and hence one would think that we need to consider separate cases if we hope to utilize the monotonicity results. But a computation shows that in fact I and J are just multiples of each other in all three cases, so it suffices to show, say, that $\lim_{R\to\infty} I=0$.

(2) Note that for $R>1$ one has

$$\begin{aligned} I_G &\leq \frac{C}{R^{4t+2}} \int_{R<|x|<2R} |x|^{\alpha(2t+1)-2t\beta} \\ &\leq \frac{C}{R^{4t+2}} R^{N+\alpha(2t+1)-2t\beta}, \end{aligned}$$

and so to show the non-existence we want to find some $0<t<2$ such that $4t+2>N+\alpha(2t+1)-2t\beta$, which is equivalent to $2t(\beta-\alpha+2)>(N+\alpha-2)$. Now recall that we are assuming that $0<N+\alpha-2<4(\beta-\alpha+2)$ and hence we have the desired result by taking $t<2$ but sufficiently close. The proof of the non-existence results for (3) and (4) are similar and we omit the details.

(5) We now assume that $N+\alpha-2>0$. In showing the existence of stable sub-/super-solutions we need to consider $\beta-\alpha+2<0$ and $\beta-\alpha+2>0$ separately.

- $(\beta-\alpha+2<0)$. Here we take $u(x)=0$ in the case of (G) and $u=1$ in the case of (L) and (M). In addition we take $g(x)=\varepsilon$. It is clear that in all cases u is the appropriate sub- or super-solution. The only thing one needs to check is the stability. In all cases this reduces to trying to show that we have

$$\sigma \int (1+|x|^2)^{\frac{\alpha}{2}-1}\phi^2 \leq \int (1+|x|^2)^{\frac{\alpha}{2}} |\nabla\phi|^2,$$

  for all $\phi\in C_c^\infty$, where σ is some small positive constant; it is either ε or $p\varepsilon$ depending on which equation we are examining. To show this we use the result from Corollary 3 and we drop a few positive terms to arrive at

$$\int (1+|x|^2)^{\frac{\alpha}{2}} |\nabla\phi|^2 \geq (t+\frac{\alpha}{2}) \int \left(N-2(t+1)\frac{|x|^2}{1+|x|^2}\right)(1+|x|^2)^{-1+\frac{\alpha}{2}},$$

  which holds for all $\phi\in C_c^\infty$ and $t,\alpha\in\mathbb{R}$. Now, since $N+\alpha-2>0$, we can choose t such that $-\frac{\alpha}{2}<t<\frac{n-2}{2}$. So, the integrand function on the right-hand side is positive, and since for small enough σ we have

$$\sigma \leq (t+\frac{\alpha}{2})(N-2(t+1)\frac{|x|^2}{1+|x|^2}) \quad \text{for all} \quad x\in\mathbb{R}^N$$

  we get stability.
- $(\beta-\alpha+2>0)$. In the case of (G) we take $u(x)=-\frac{\beta-\alpha+2}{2}\ln(1+|x|^2)$ and $g(x):=(\beta-\alpha+2)(N+(\alpha-2)\frac{|x|^2}{1+|x|^2})$. By a computation one sees that u is a sub-solution of (G), and hence we now need only to show the stability, which amounts to showing that

$$\int \frac{g(x)\psi^2}{(1+|x|^2)^{-\frac{\alpha}{2}+1}} \leq \int \frac{|\nabla\psi|^2}{(1+|x|^2)^{-\frac{\alpha}{2}}},$$

for all $\psi \in C_c^\infty$. To show this we use Corollary 3. So we need to choose an appropriate t in $-\frac{\alpha}{2} \leq t \leq \frac{N-2}{2}$ such that for all $x \in \mathbb{R}^N$ we have

$$\begin{aligned}(\beta-\alpha+2)\left(N+(\alpha-2)\frac{|x|^2}{1+|x|^2}\right) \quad &\leq \quad (t+\frac{\alpha}{2})^2\frac{|x|^2}{(1+|x|^2} \\ &+ (t+\frac{\alpha}{2})\left(N-2(t+1)\frac{|x|^2}{1+|x|^2}\right).\end{aligned}$$

With a simple calculation one sees that we just need to have

$$\begin{aligned}(\beta-\alpha+2) \quad &\leq \quad (t+\frac{\alpha}{2}), \\ (\beta-\alpha+2)\left(N+\alpha-2\right) \quad &\leq \quad (t+\frac{\alpha}{2})\left(N-t-2+\frac{\alpha}{2})\right).\end{aligned}$$

If one takes $t=\frac{N-2}{2}$ in the case where $N \neq 2$ and t close to zero in the case for $N=2$, one easily sees that the above inequalities both hold after considering all the constraints on α, β and N.

We now consider the case of (L). Here one takes $g(x) := \frac{\beta-\alpha+2}{p-1}(N + (\alpha-2-\frac{\beta-\alpha+2}{p-1})\frac{|x|^2}{1+|x|^2})$ and $u(x)=(1+|x|^2)^{-\frac{\beta-\alpha+2}{2(p-1)}}$. Using essentially the same approach as in (G) one shows that u is a stable sub-solution of (L) with this choice of g.

For the case of (M) we take $u(x)=(1+|x|^2)^{\frac{\beta-\alpha+2}{2(p+1)}}$ and $g(x) := \frac{\beta-\alpha+2}{p+1}(N+ (\alpha-2+\frac{\beta-\alpha+2}{p+1})\frac{|x|^2}{1+|x|^2})$. □

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Department of Mathematics, Stanford University, Stanford, California 94305
E-mail address: ctcowan@stanford.edu

Department of Mathematics, University of British Columbia, Vancouver, B.C., Canada V6T 1Z2
E-mail address: fazly@math.ubc.ca