2.3 (Continued)

① Example L
of function

② graph & pict
limit exists

Direct Substitution

If f a polynomial or a rational function & a $(x=a)$ in the domain of f, then

$$\lim_{x \to a} f(x) = f(a).$$

Ex Compute the limit, !

① $\lim_{x \to 3} (x^2+6x+9)$.

② $\lim_{x \to 3} \dfrac{x^2-6x+2}{x-4}$

③ $\lim_{x \to 1} \dfrac{x^2-1}{x-1}$.

solution

1. $f(x) = \cancel{f(x)} x^2 + 6x + 9$ a polynomial

hence

$$\lim_{x \to 3} (x^2 + 6x + 9) = f(3) = 3^2 + 6 \cdot 3 + 9$$
$$= 9 + 18 + 9$$
$$= 36.$$

2. $f(x) = \dfrac{x^2 - 6x + 2}{x - 4}$ a rational function & $a = 3$ is in the domain of f (i.e bottom not zero)

hence

$$\lim_{x \to 3} f(x) = f(3) = \frac{3^2 - 6 \cdot 3 + 2}{3 - 4}$$

$$= \frac{9 - 18 + 2}{-1} = \frac{-7}{-1} = 7.$$

3 $f(x) = \frac{x^2-1}{x-1}$ $\lim_{x \to 1} f(x)$.

Cannot ~~Apply Direct subst~~

f(x) a rational function but

$a=1$ is not in domain of f

since bottom is zero; so

cannot Apply direct substitution.

Cannot Apply limit law

$$\lim_{x \to 1} \frac{x^2-1}{x-1} = \frac{\lim_{x \to 1} x^2-1}{\lim_{x \to 1} (x-1)}$$

since $\lim_{x \to 1} (x-1) = 0$.

Aside

Rule If $f(x) = g(x)$ for all $x \neq a$ then

$$\lim_{x \to a} f(x) = \lim_{x \to a} g(x).$$

Return to $f(x) = \dfrac{x^2 - 1}{x - 1}$. Then

$$f(x) = \frac{(x+1)(x-1)}{x-1} = x + 1 = g(x) \quad \text{for } x \neq$$

so $\displaystyle \lim_{x \to 1} f(x) = \lim_{x \to 1} g(x) = \lim_{x \to 1} (x+1) = 2$

Ex Compute

$$\lim_{h \to 0} \frac{(h+2)^2-4}{h}.$$

Solution Cannot apply limit laws or substitution; need to simplify before taking limit.

$h \neq 0$

$$f(h) = \frac{(h+2)^2-4}{h}$$

$$= \frac{h^2+4h+4-4}{h} = \frac{h^2+4h}{h} = h+4$$

$$\lim_{h \to 0} f(h) = \lim_{h \to 0} (h+4) = 4.$$

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Compute $\lim\limits_{t \to 0} \dfrac{\sqrt{t^2+9}-3}{t^2}$.

sol. Cannot Apply limit Laws.

Trick

$$\left(\frac{\sqrt{t^2+9}-3}{t^2}\right)\underbrace{\left(\frac{\sqrt{t^2+9}+3}{\sqrt{t^2+9}+3}\right)}_{=1}$$

$$= \frac{(t^2+9)-9}{t^2\left[\sqrt{t^2+9}+3\right]} = \frac{1}{\sqrt{t^2+9}+3}$$

SO

$$\lim_{t\to 0}\frac{\sqrt{t^2+9}-3}{t^2} = \lim_{t\to 0}\frac{1}{\sqrt{t^2+9}+3}$$

$$= \frac{1}{\lim_{t\to 0}\left[\sqrt{t^2+9}+3\right]}$$

$$= \frac{1}{\sqrt{9}+3} = \frac{1}{6}.$$

<u>Ex</u> Prove ~~that~~ (see Ex 8 book)

$\lim_{x\to 1}\frac{x-1}{|x-1|}$ does not exist.

<u>sol.</u> write $|x-1|$ as piecewise & ~~then~~ Always do left & right limits

$$|x-1| = \begin{cases} x-1 & \text{if } x-1 \geq 0 \\ -(x-1) & \text{if } x-1 < 0 \end{cases}$$

$$= \begin{cases} x-1 & x \geq 1 \\ 1-x & x < 1. \end{cases}$$

$$\lim_{x \to 1^+} \frac{x-1}{|x-1|} = \lim_{x \to 1^+} \frac{x-1}{x-1} = 1$$

$$\lim_{x \to 1^-} \frac{x-1}{|x-1|} = \lim_{x \to 1^-} \frac{x-1}{1-x} = -1.$$

since left & right limits <u>not</u> equal we see limit does not exist.

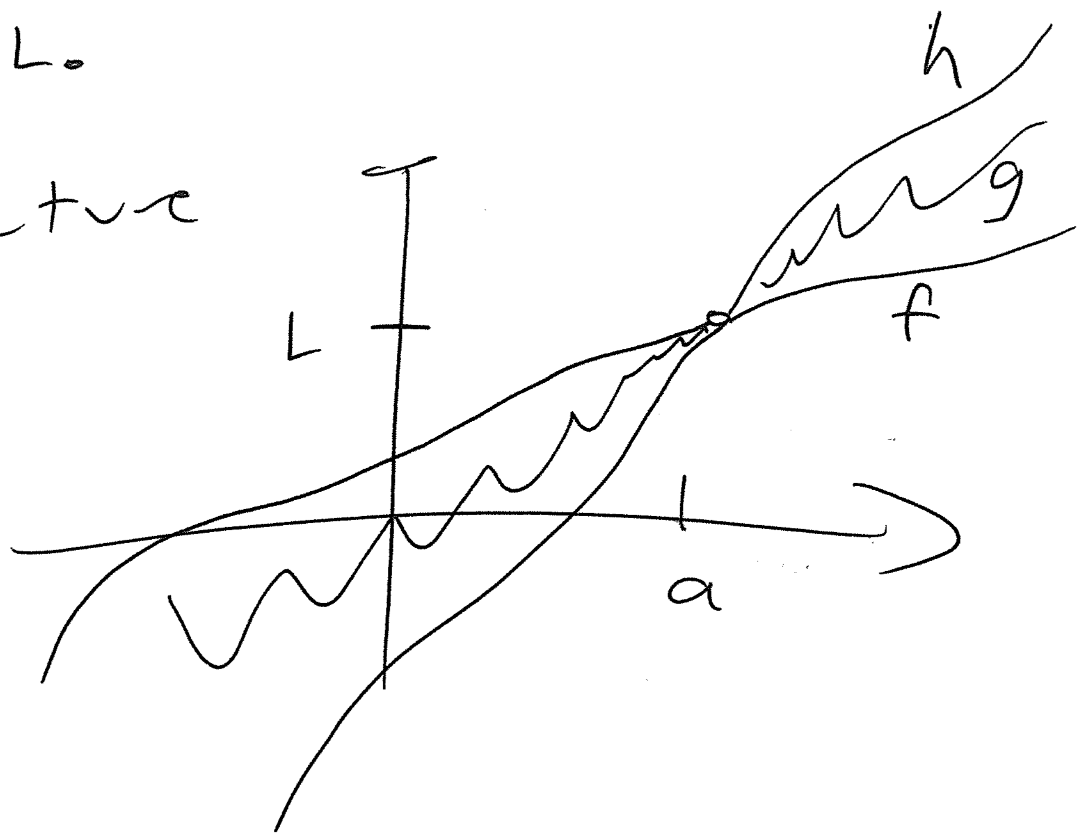
SQUEEZE Th^m

If $f(x) \le g(x) \le h(x)$ all x near a (except possibly at a) &

$\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$. Then

$\lim_{x\to a} g(x) = L$.

picture



~~Ex Compt~~

Ex Compute

$$\lim_{x\to 1} (x-1)^2 \sin\left(\frac{1}{(x-1)}\right).$$

solution note $\lim_{x\to 1} \sin\left(\frac{1}{x-1}\right)$

does not exist; so cannot

Apply limit Laws.

But note $-1 \le \sin\left(\frac{1}{x-1}\right) \le 1$

all $x \ne 1$. since $(x-1)^2 \geq 0$

we see

$$-1(x-1)^2 \le (x-1)^2 \sin\left(\frac{1}{x-1}\right) \le (x-1)^2 \quad \text{all } x \ne 1$$

$f(x) = -(x-1)^2$

$h(x) = (x-1)^2$

so $\lim_{x\to 1} f(x) = \lim_{x\to 1} h(x) = 0$

so by SQUEEZE thm

$$\lim_{x \to 1} (x-1)^2 \sin\left(\frac{1}{x-1}\right) = 0.$$

Ex ~~Compute~~ Let $f(x) = \begin{cases} x^2+x-1 & \text{if } x < 2 \\ \sqrt{x^4+9} & \text{if } x \geq 2. \end{cases}$

~~⊕ dip~~

compute

(1) $\lim_{x \to 3} f(x)$

~~(3) $\lim_{x \to 2}$~~ (2) $\lim_{x \to 2} f(x)$ exist?

2.5 Continuity (Add more graphs)

A function $f(x)$ is continuous if to compute a limit one can just plug number in.

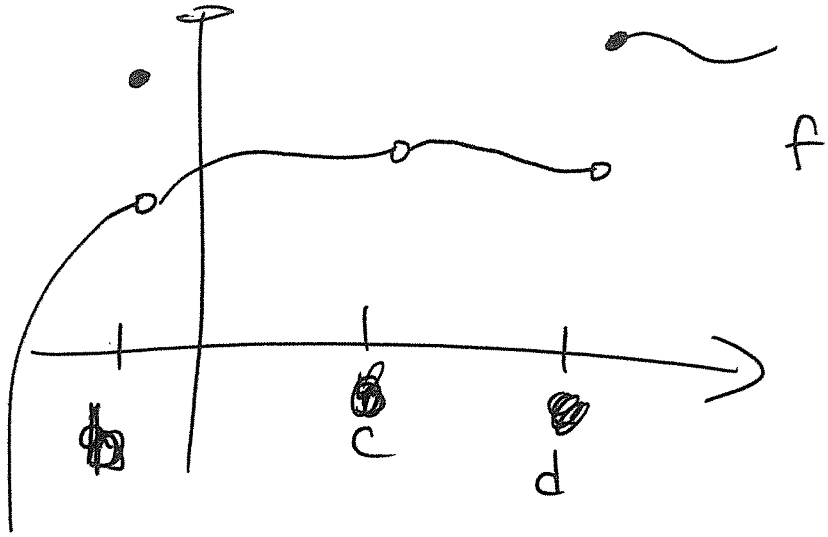
Formally f is continuous at a if $\lim_{x \to a} f(x) = f(a)$ & we say

f continuous if f continuous at all a in the domain of f.

If f is not continuous at a we say f discontinuous.

f can be discontinuous at a if

- f not defined at a
- $\lim_{x \to a} f(x)$ does not exist
- $\lim_{x \to a} f(x) = L$ but $L \neq f(a)$.



~~f continuous on its domain except~~

potential problem spots

$x = b$, $x = c$, $x = d$.

f not continuous at $x = b$ because

$$\underbrace{f(b)}_{\text{wrong value}} \neq \underbrace{\lim_{x \to b} f(x)}_{\text{exists}}$$

f not defined at $x = c$ hence
f not continuous at c.

f not continuous at d
since $\lim_{x \to d} f(x)$ does not exist.

Also note f continuous at all x
except $x = \cancel{b}, c, d$.

<u>Ex</u> where are each of the following function discontinuous?

a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

b) $f(x) = \begin{cases} \frac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$

c) $f(x) = \begin{cases} \frac{x^2 - x - 2}{x - 2} & x \neq 2 \\ 1 & x = 2 \end{cases}$

[14.1]

Ex show $f(x) = \dfrac{x^2+1}{x-1}$ is

continuous at $a = 3$.

Defn. f is continuous on an interval

if it is continuous at each number in the interval;

if the interval closed $[a,b]$

just need to check

$\lim_{x \to a^+} f(x) = f(a)$ & $\lim_{x \to b^-} f(x) = f(b)$

for end points (of course can't compute limit at a from left).

Thm If f, g are continuous at a & $c \in \mathbb{R}$ then

1. $f \pm g$ is continuous at a
2. cf is " " "
3. fg is " " "

(4) $\frac{f}{g}$ is continuous at a provided $g(a) \neq 0$.

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Thm - Any polynomial is continuous on $\mathbb{R}$ (ie continuous at all $a \in \mathbb{R}$)

- ~~Any~~ Any rational function is continuous at all a in the domain the function.

Ex where are each of the following functions discontinuous?

(a) $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

(b) $f(x) = \begin{cases} \dfrac{1}{x^2} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0. \end{cases}$

Ex Is
$$f(x) = \begin{cases} \dfrac{(x+3)(x+1)}{x+2} & \text{if } x \le 0 \\[2ex] 3x^2 + x + \dfrac{3}{2} & \text{if } x > 0 \end{cases}$$

continuous on $[-1,1]$?

Ex what value of c makes

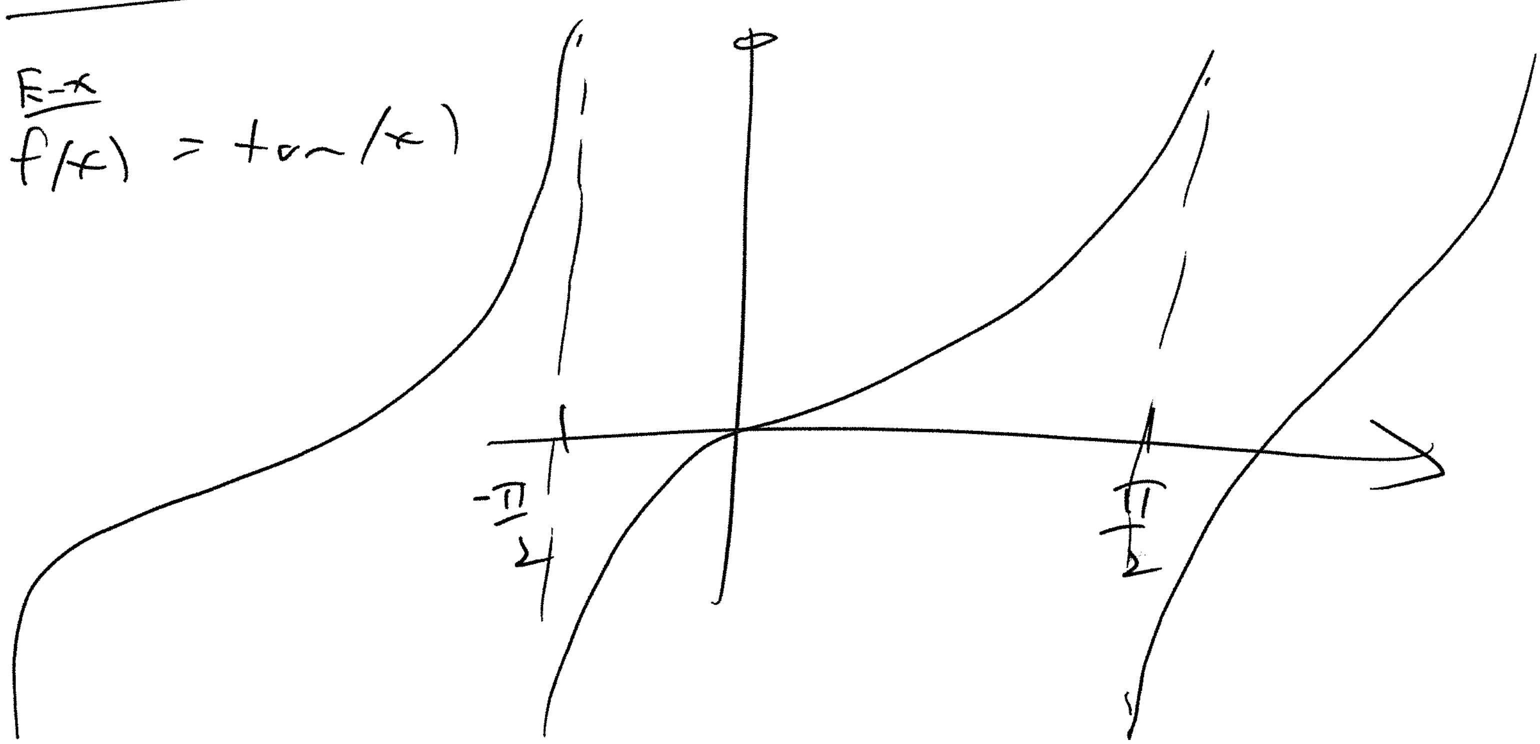
$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x-2} & \text{if } x \neq 2 \\[2ex] c & \text{if } x = 2 \end{cases}$$

continuous on $\mathbb{R}$?

Thm 7 The following functions are continuous at each point in their domains

- polynomials
- rational functions
- root functions
- trig functions
- exponential functions
- power functions.

<u>Ex</u>
$f(x) = \tan(x)$



But note $x = -\frac{\pi}{2}, \frac{\pi}{2}, \ldots$

not in domain of $f(x)$.

~~Thm If f continuous at b & $\lim_{x \to a} g(x) = b$, then~~

$\cancel{\lim_{x \to a} f[g(x)] = f(b)}$ ~~ie~~

$\cancel{\lim_{x \to a} f(g(x)) = f[\lim_{x \to a} g(x)]}$.

Thm If g continuous at a & f continuous at $g(a)$ then

$(f \circ g)(x) = f[g(x)]$ is continuous at a.

Ex 9 where are the functions

① $h(x) = \sin(x^2)$

② ~~$F(x) = e^{\frac{1+\cos(x)}{x}}$~~

$F(x) = e^{\frac{\cos(x)}{x}}$ continuous?

Start Here

The intermediate value theorem.

Suppose f is continuous on $[a,b]$ & N is any number between $f(a)$ & $f(b)$. Then there is some $c \in [a,b]$ such that

$f(c) = N$.

(In practice will use $N=0$ mostly)