Section 2.8

Ex If $f(x) = \sqrt{x}$ find the derivative of f. State the domain of f'.

~~Section 2.8~~

~~Ex Compute $f'(x)$ when $f(x)=\sqrt{x}$~~

Solution

$$\frac{f(x+h)-f(x)}{h} = \frac{\sqrt{x+h}-\sqrt{x}}{h}$$

$$= \frac{\left(\sqrt{x+h}-\sqrt{x}\right)\left(\sqrt{x+h}+\sqrt{x}\right)}{h\left[\sqrt{x+h}+\sqrt{x}\right]}$$

$$= \frac{(x+h)-x}{h\left(\sqrt{x+h}+\sqrt{x}\right)}$$

$$= \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

so for $x > 0$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{2}{\sqrt{x}}.$$

So domain of f' is $(0, \infty)$.

<u>other notations for derivaties</u>

If $y = f(x)$

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx} f(x).$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \left.\frac{d}{dx} f\right|_{x=1} = f'(1).$$

L3

Higher order derivatives.

given f a function then

If f' exists we can take

Another derivative

$$f''(x) := (f')'(x)$$

$$f'''(x) = (f'')'(x)$$

$$f^{(4)}(x) = f''''(x).$$

$$y = f(x),$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=4} = f''(4)$$

$$\left.\frac{d^3y}{dx^2}\right|_{x=4} = f'''(4).$$

Recall if $d(t)$ was ~~dist~~ distance as a function of time then

$v(t) := d'(t)$ (velocity)

we denote

$a(t) = v'(t)$ (acceleration

$= (d'(t))'$

$= d''(t)$

Section 3.1

Derivatives of polynomial & exponential functions.

(CANNOT use these rules on test; get zero)

Rules

Rule 1 If $f(x) = c$ (constant)

then $f'(x) = 0$. $\left(\frac{d}{dx} c = 0\right)$.

Rule 2 (power rule) If $n \in \mathbb{R}$ ~~...~~ ~~...~~

$f(x) = x^n$ ~~...~~ ($n \neq 0$ $n=0$ just constant function

& $f'(x) = n x^{n-1}$ $\left(\frac{d}{dx} x^n = n x^{n-1}\right)$.

Rule 3

If $f(x)$ & $g(x)$ are functions & $a, b \in \mathbb{R}$ then

$$(a f(x) + b g(x))' = a f'(x) + b g'(x)$$

or

$$\frac{d}{dx}\left(a f(x) \pm b g(x)\right) = a \frac{df}{dx} \pm b \frac{dg}{dx}.$$

Ex Compute $f'(x)$ if $f(x) = \frac{1}{\sqrt[3]{x}}$.

solution

write as a power function with power on top; ie

$$f(x) = \frac{1}{\sqrt[3]{x}} = \frac{1}{x^{\frac{1}{3}}} = x^{-\frac{1}{3}}.$$

now Apply power rule;

$$f'(x) = \left(-\frac{1}{3}\right) x^{-\frac{1}{3}-1} = -\frac{1}{3} x^{-\frac{4}{3}}$$

(on test no need to simplify,

Ex Compute ~~the derivative of~~ ~~$\frac{d}{dx}$~~ $f'(x)$ if $f(x) = x(\sqrt{x} + 4x)$.

solution.

$$f(x) = x(x^{\frac{1}{2}} + 4x) = x^{\frac{3}{2}} + 4x^2$$

$$f'(x) = \frac{3}{2} x^{\frac{3}{2}-1} + 4(2)x^{2-1}$$

$$= \frac{3}{2} x^{\frac{3}{2}-1} + 4(2)x.$$

Ex 6 pg 177

Find the points on the curve $y = x^4 - 6x^2 + 4$ where the tangent line is Horizontal?

L8

Ex 7 the equation of motion of a particle is $s = 2t^3 - 5t^2 + 3t + 4$ where s in centimeters & t in seconds. Find the Accelartion as a function of time & find ~~[illegible]~~. the Acceleration After 2 seconds.

$S(t)$ - distance As a function of time

$v(t) = s'(t)$ velocity

$a(t) = v'(t) = s''(t)$ Acceleration.

$s'(t) = 6t^2 - 10t + 3$

$s''(t) = 12t - 10$

$\boxed{a(t) = 12t - 10}$

$a(2) = 24 - 10$

$= 14 \frac{cm}{s^2}$

Ex Find the ~~eqn~~ equation of the line tangent to $f(x) = x^3 - 1$ at the point $(2, 7)$.

10

Proofs you must know (see webpage)

$(cf)' = c f'$ (1)

$(f+g)' = f' + g'$ (2)

fg some others.

proof of (2)

$$(f+g)'(x) = \lim_{h\to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h}$$

$$= \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h} + \frac{g(x+h)-g(x)}{h}\right]$$

$$= \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) + \lim_{h\to 0}\left[\frac{g(x+h)-g(x)}{h}\right]$$

$$= f'(x) + g'(x).$$

Exponential functions

$f(x) = a^x$

$$f'(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$= \lim_{h\to 0}\left[\frac{a^{x+h}-a^x}{h}\right]$$

$$= \lim_{h\to 0}\left(\frac{a^x\cdot a^h - a^x}{h}\right)$$

$$= \lim_{h\to 0}\left[\underbrace{a^x}_{\text{constant with respect to } h}\frac{(a^h-1)}{h}\right]$$

$$= a^x \lim_{h\to 0}\frac{(a^h-1)}{h} = a^x\lim_{h\to 0}\left(\frac{a^{h+0}-a^0}{h}\right)$$

$$= a^x\, f'(0).$$

Definition of e.

e is the # such that

$$\lim_{h \to 0} \left(\frac{e^h - 1}{h} \right) = 1 \qquad \Big(e \approx 2.71 \ldots$$

~~[illegible]~~ So if

$f(x) = e^x$ then

$f'(x) = e^x \quad f'(0)$

$$= e^x \lim_{h \to 0} \frac{e^h - 1}{h} = e^x$$

So $\boxed{\dfrac{d}{dx} e^x = e^x}$

$f(x) = e^x \qquad f'(x) = e^x$

Using base e makes derivatives easier.

Ex compute $\frac{d}{dx}(2e^x + 7x^2)$

sol.

$$\frac{d}{dx}(2e^x + 7x^2) = 2\frac{d}{dx}(e^x) + 7\frac{d}{dx}x^2$$
$$= 2e^x + 7(2)x.$$

Ex 9 pg 179 At what point on the curve $y = e^x$ is the tangent line parrallel to the line $y = 2x$.

3.2 Product & Quotient rule

Product rule.

If $f'(x)$ & $g'(x)$ exist then

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$

or $$\frac{d}{dx}(fg) = \left(\frac{df}{dx}\right)g + f\left(\frac{d}{dx}g\right)$$

Ex If $f(x) = e^x(3x^2+x)$

compute $f'(x)$.

solution

$$f'(x) = \frac{d}{dx}\left(e^x(3x^2+x)\right)$$

$$= \left(\frac{d}{dx}e^x\right)(3x^2+x)$$

$$+ e^x \frac{d}{dx}(3x^2+x)$$

$= e^x (3x^2 + x)$

$+ e^x (3/2) x + 1)$.

Quotient rule.

If $f'(x)$ & $g'(x)$ exist & $g(x) \neq 0$

then

$$\frac{d}{dx}\left(\frac{f(x)}{g(x)}\right) = \frac{f'(x)g(x) - f(x)\,g'(x)}{(g(x))^2}.$$

Ex Compute $h'(x)$ if $h(x) = \frac{1}{x^{\frac{1}{2}}}$

using power rule & quotient rule

① power rule

$h(x) = x^{-\frac{1}{2}}$ $\qquad h'(x) = -\frac{1}{2} x^{-\frac{1}{2}-1}$

$= -\frac{1}{2} x^{-\frac{3}{2}}$

---

2) Quotient rule.

$h(x) = \frac{1}{x^{\frac{1}{2}}} = \frac{f(x)}{g(x)}$ $\qquad f(x) = 1$

$g(x) = x^{\frac{1}{2}}$

$h'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$ $\qquad f'(x) = 0$

$g'(x) = \frac{1}{2} x^{\frac{1}{2}-1}$

$= \frac{1}{2} x^{-\frac{1}{2}}$

$= \frac{-1 \, \frac{1}{2} x^{-\frac{1}{2}}}{(x^{\frac{1}{2}})^2} = -\frac{1}{2x^{\frac{3}{2}}}.$

NEED to do lots of practice problems with product & quotient rules.

Ex $y = \dfrac{\sqrt{x}}{2+x}$ $\qquad \dfrac{dy}{dx} = ?$

Ex $v(t) = \dfrac{4+t}{te^t}$ $\qquad v'(t) = ?$

Ex $f(x) = \dfrac{x}{x^2-1}$ $\qquad f'(x) = ?$

Section 3.3

Derivatives of trig functions.

Keypoint (memorize)

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

more important

$$\boxed{\lim_{\theta \to 0} \frac{\cos(\theta)-1}{\theta} = 0}$$

less important

$$\sin(\alpha+\beta) = \sin(\alpha)\cos(\beta) + \cos(\alpha)\sin(\beta)$$

$$\cos(\alpha+\beta) = \cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta).$$

Rule. $\frac{d}{dx}\sin(x) = \cos(x)$

proof $f(x) = \sin(x)$.

$$\frac{f(x+h)-f(x)}{h} = \frac{\sin(x+h)-\sin(x)}{h}$$

$$= \frac{1}{h}\left[\sin(x)\cos(h)+\cos(x)\sin(h)-\sin(x)\right]$$

$$= \frac{\sin(x)\left[\cos(h)-1\right]}{h} + \cos(x)\frac{\sin(h)}{h}$$

$$\frac{d}{dx}\sin(x) = f'(x) = \lim_{h\to 0}\left[\text{''} \qquad \text{''}\right]$$

$$= \lim_{h\to 0} \sin(x)\left[\frac{\cos(h)-1}{h}\right] + \lim_{h\to 0}\cos(x)\frac{\sin(h)}{h}$$

$$= \sin(x)\lim_{h\to 0}\left(\frac{\cos(h)-1}{h}\right) + \cos(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \sin(x)\cdot 0 + \cos(x)\cdot 1 = \cos(x).$$

L20

Rules

$$\frac{d}{dx}\sin(x) = \cos(x)$$

$$\frac{d}{dx}\cos(x) = -\sin(x)$$

Proof of $\frac{d}{dx}\cos(x) = -\sin(x)$.

proof

$f(x) = \cos(x)$

$$\frac{f(x+h)-f(x)}{h} = \frac{1}{h}\left[\cos(x+h) - \cos(x)\right]$$

$$= \frac{1}{h}\left[\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)\right]$$

$$= \cos(x)\frac{[\cos(h) - 1]}{h} - \sin(x)\frac{\sin(h)}{h}$$

$$f'(x) = \lim_{h\to 0} \cos(x)\frac{(\cos(h)-1)}{h}$$

$$-\sin(x)\lim_{h\to 0}\frac{\sin(h)}{h}$$

$$= \cos(x)\cdot 0 - \sin(x)\cdot 1 = -\sin(x).$$

Rule $\frac{d}{dx}\tan(x) = \sec^2(x) \quad \left(= \frac{1}{\cos^2(x)}\right).$

proof $h(x) = \tan(x) = \frac{\sin(x)}{\cos(x)}$

$$\frac{d}{dx}\tan(x) = \frac{\left(\frac{d}{dx}\sin(x)\right)\cos(x) - \left(\frac{d}{dx}(\cos(x))\right)\sin(x)}{\cos^2(x)}$$

$$= \frac{\cos^2(x) - (-\sin(x))\sin(x)}{\cos^2(x)}$$

$$= \frac{\sin^2(x) + \cos^2(x)}{\cos^2(x)} = \frac{1}{\cos^2(x)}$$

$$= \sec^2(x).$$

$\csc(x) := \frac{1}{\sin(x)}$

$\sec(x) := \frac{1}{\cos(x)}$

$\cot(x) := \frac{1}{\tan(x)}$

Then

$\frac{d}{dx} \csc(x) = -\csc(x)\cot(x)$

$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$

$\frac{d}{dx}(\cot(x)) = -\csc^2(x).$

I would memorize

100% $\left(\begin{array}{l} \frac{d}{dx} \sin(x) = \cos(x) \\ \frac{d}{dx} \cos(x) = -\sin(x). \end{array} \right.$

Maybe memorize

$$\frac{d}{dx}\tan(x) = \sec^2(x).$$

The others you can work out if ~~need~~ need be (but you must remember definition of

$\csc(x)$
$\sec(x)$
$\cot(x)$)

Ex Compute $h'(x)$ if

$$h(x) = \frac{\sin(x)}{x^2}.$$

Ex compute $\frac{d}{dx}\left(x^2 \cos(x)\right)$.

more examples.

This limit

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$ is quite

often used to compute other

limits.

Ex Evaluate $\displaystyle\lim_{x \to 0} \frac{\pi \sin(x^2)}{x}$.

Ex Evaluate $\displaystyle\lim_{x \to 2} \frac{\sin(2x-4)}{\pi(x-2)}$

Ex $\displaystyle\lim_{x \to 0} \frac{\sin(7x)}{4x}$.