Last Class (in 3.6) [1]

$$\frac{d}{dx}\ln(x) = \frac{1}{x}$$

Proofs you need to know.

Rule

$$\frac{d}{dx} b^x = x\ln(b).$$

Proof write $y = b^x$ (need to give function a name)

Now ln both sides, so

$$\ln(y) = \ln(b^x) = x\ln(b)$$ by ln rule

Now take $\frac{d}{dx}$ of both sides

& make to recall $y = y(x)$

(so use chain rule)

2

$$\frac{d}{dx}\left(\ln(y)\right) = \frac{d}{dx}\left(x\ln(b)\right)$$

$$\frac{1}{y}\,y' = \ln(b)$$

so $\dfrac{dy}{dx} = y\ln(b)$

but write Answer in terms of x

$$\frac{dy}{dx} = b^x \ln(b)$$

so $\boxed{\dfrac{d}{dx}\, b^x = b^x \ln(b)}$

Trick above has name

Logarithmic differentiation

Compute $\frac{d}{dx}\left(x^{\sin(x)}\right)$.

Cannot Apply rules to compute. Again give it a name

say

$$y = x^{\sin(x)}$$

now ln both sides

$$\ln(y) = \ln\left(x^{\sin(x)}\right) = \sin(x)\ln(x).$$

So

$$\frac{1}{y}\frac{dy}{dx} = \cos(x)\ln(x) + \frac{\sin(x)}{x}$$

$$\Rightarrow \frac{dy}{dx} = y\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right]$$

Write in terms of x.

$$\frac{d}{dx}\left(x^{\sin(x)}\right) = x^{\sin(x)}\left[\cos(x)\ln(x) + \frac{\sin(x)}{x}\right]$$

<u>Rule</u>. $\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x\ln(b)}$.

<u>proof</u>. Again given $\log_b(x)$ a name

$y = \log_b(x)$ (no clue how to take derivative)

recall

$y = \log_b(x) \iff b^y = x$

so compute $\frac{dy}{dx}$ from

5

$$1 = \frac{d}{dx}\left(b^y\right) = \ln(b)\, b^y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\ln(b)\, b^y} \overset{\text{write in terms of } x}{=} \frac{1}{\ln(b)\, x}$$

So $$\frac{d}{dx}\left(\log_b(x)\right) = \frac{1}{x\ln(b)}$$

Key point: Compute $\frac{d}{dx} f^{-1}(x)$.

Give it a name

$y = f^{-1}(x)$

but confused with f^{-1}

$$y = f^{-1}(x) \iff f(y) = x$$

so compute $\frac{dy}{dx}$ from

$$1 = f'(y)\,\frac{dy}{dx}$$

or
$$\frac{dy}{dx} = \frac{1}{f'(y)} = \frac{1}{f'[f^{-1}(x)]}$$

write in terms of x

or
$$\boxed{\frac{d}{dx}\left(f^{-1}(x)\right) = \frac{1}{f'(f^{-1}(x))}}$$

Another Application of Logarithmic derivatives.

Before needed it to compute

$\frac{d}{dx} \sin(x)^{\ln(x)}$. But it can also make computations easier.

Let

$$f(x) = \frac{(x-1)(x-2)^2(x-3)^3}{\cancel{\ldots}\,\cdot \cos(x)}$$

Find $f'(x)$.

~~ln both sides.~~

L8

let $f(x) = \dfrac{(x-1)(x-2)^2(x^2+3)}{\cos(x)}$.

Compute $f'(x)$.

ln both sides

$$\ln(f(x)) = \ln(x-1) + 2\ln(x-2) + \ln(x^3+3)$$
$$- \ln(\cos(x)).$$

so

$$\frac{1}{f(x)} f'(x) = \frac{1}{x-1} + \frac{2}{x-2} + \frac{3x^2}{x^3+3}$$
$$- \frac{1}{\cos(x)}(-\sin(x))$$

9

So

$$f'(x) = \frac{(x-1)(x-2)^2(x^3+3)}{\cos(x)}\left[\frac{1}{x-1} + \frac{2}{x-2} + \frac{3x^2}{x^3+3} + \frac{\sin(x)}{\cos(x)}\right]$$

3.5 (Derivatives of inverse trig.) 10

$$\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1-x^2}}$$

(others)

$$\frac{d}{dx}\left(\cos^{-1}(x)\right) = \frac{-1}{\sqrt{1-x^2}}$$

pg 214

$$\frac{d}{dx}\left(\tan^{-1}(x)\right) = \frac{1}{1+x^2}$$

---

proof (Again proof is important here)

$\frac{d}{dx}\left(\sin^{-1}(x)\right)$.

<u>step 1</u> give it a name

$y = \sin^{-1}(x) \Leftrightarrow \sin(y) = x.$

<u>Step 2</u> Apply Implicit derivative both sides of

$$1 = \cos(y)\frac{dy}{dx}$$

$$\boxed{\frac{dy}{dx} = \frac{1}{\cos(y)}}$$

NEED to find $\cos(y)$ in terms of x.

could write

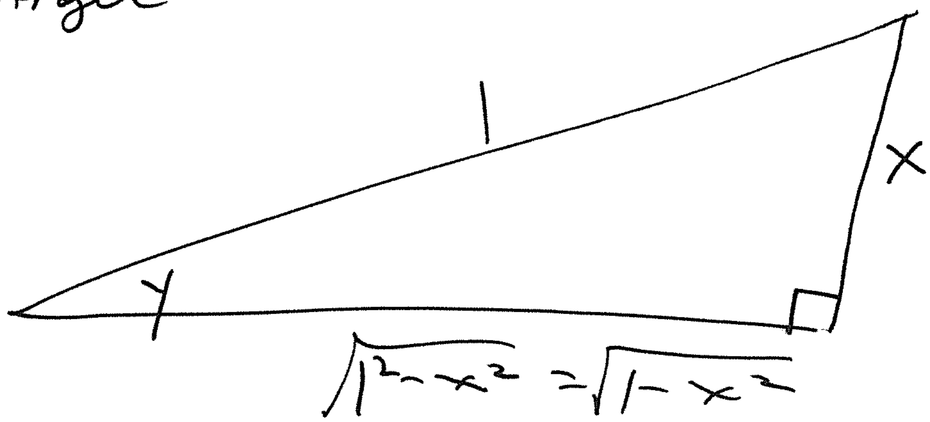
$\cos(y) = \cos[\sin^{-1}(x)]$

but we can do better.

$$\sin(y) = x$$

$$\sin(y) = \frac{x}{1}$$

Angle



So $\cos(y) = \dfrac{\sqrt{1 - x^2}}{1}$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1 - x^2}}$$

$$\boxed{\frac{d}{dx}\left(\sin^{-1}(x)\right) = \frac{1}{\sqrt{1 - x^2}}.}$$

(previous stuff all ~~[illegible]~~ on ~~the~~ test) L13

4.1 Max & min values

Today we compute max & min values of a function on its Domain

<u>Defn.</u> let $c \in D$ (D the domain of some function f).

Then $f(c)$ is

- <u>absolute maximum</u> value of f on D if $f(c) \geq f(x)$ all $x \in D$.

(write absolute max)

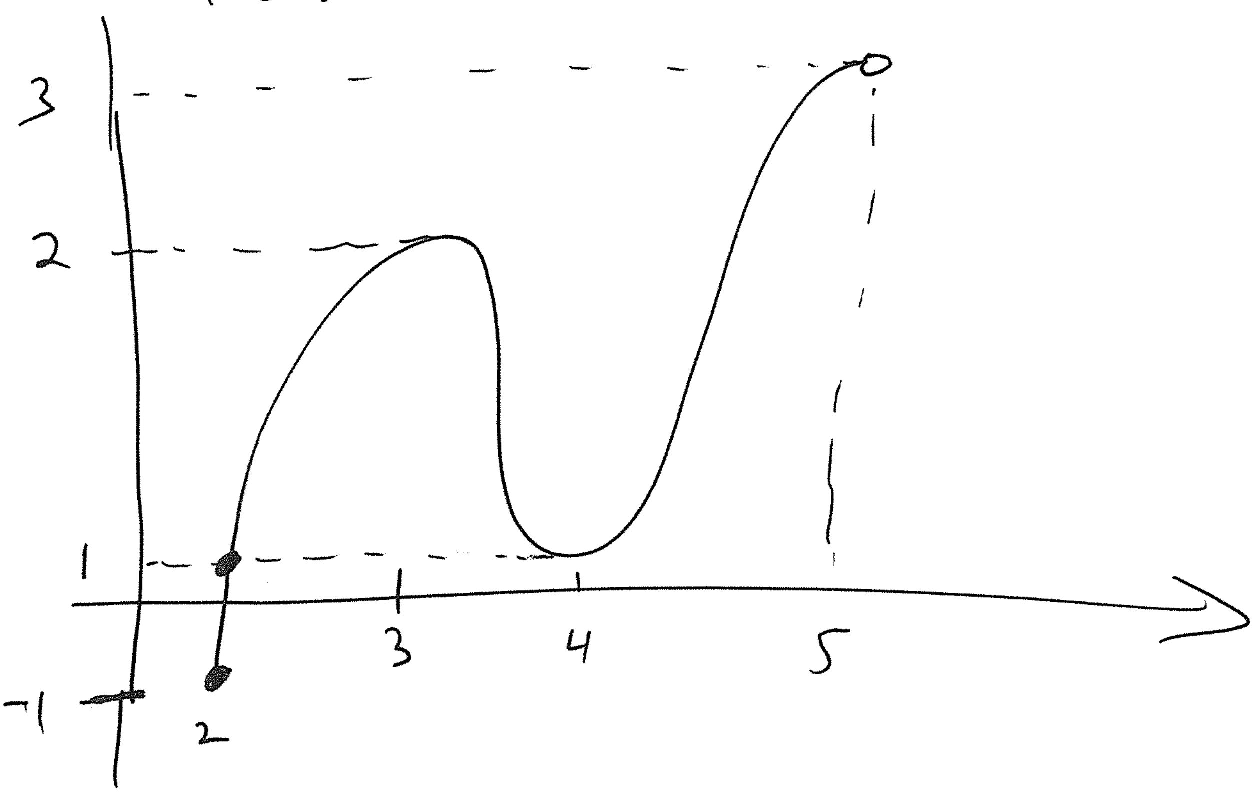
- absolute minimum (absolute min) of f on D if

$f(c) \leq f(x)$ all $x \in D$

<u>Defn.</u> let $c \in D$. then the number $f(c)$ is a

- <u>local maximum</u> (local max) of f if $f(c) \geq f(x)$ all $x \in D$ <u>near</u> c.

- <u>local minimum</u> (local min) of f if $f(c) \leq f(x)$ all $x \in D$ near c.

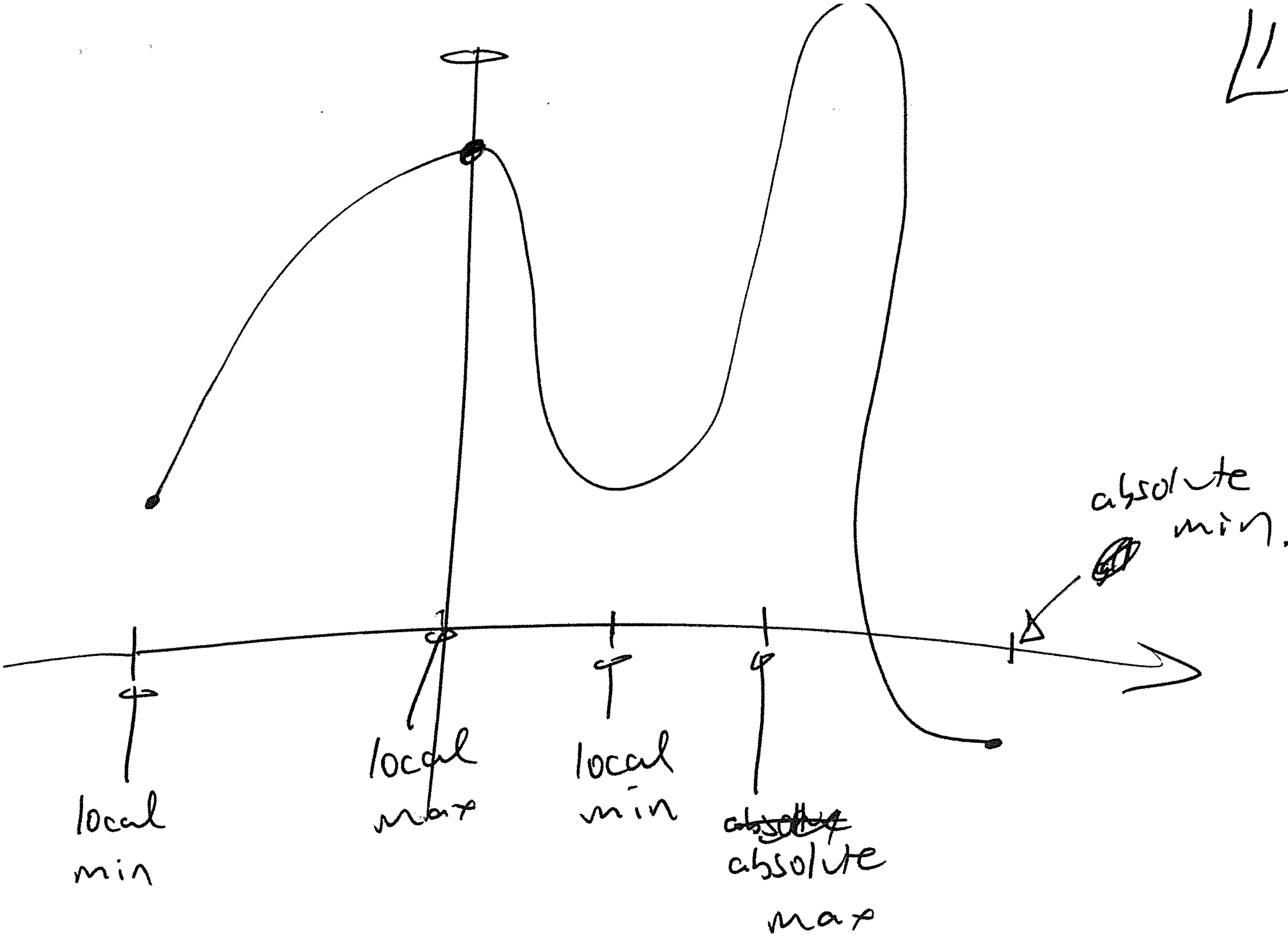


then $f(x)$ has an absolute min value of -1,

(we can say $f(x)$ has an absolute minimum value of -1 at $x = 2$).

- $f(x)$ has a local max value of 2 at $x=3$
- $f(x)$ has a local min value of 1 at $x=4$.

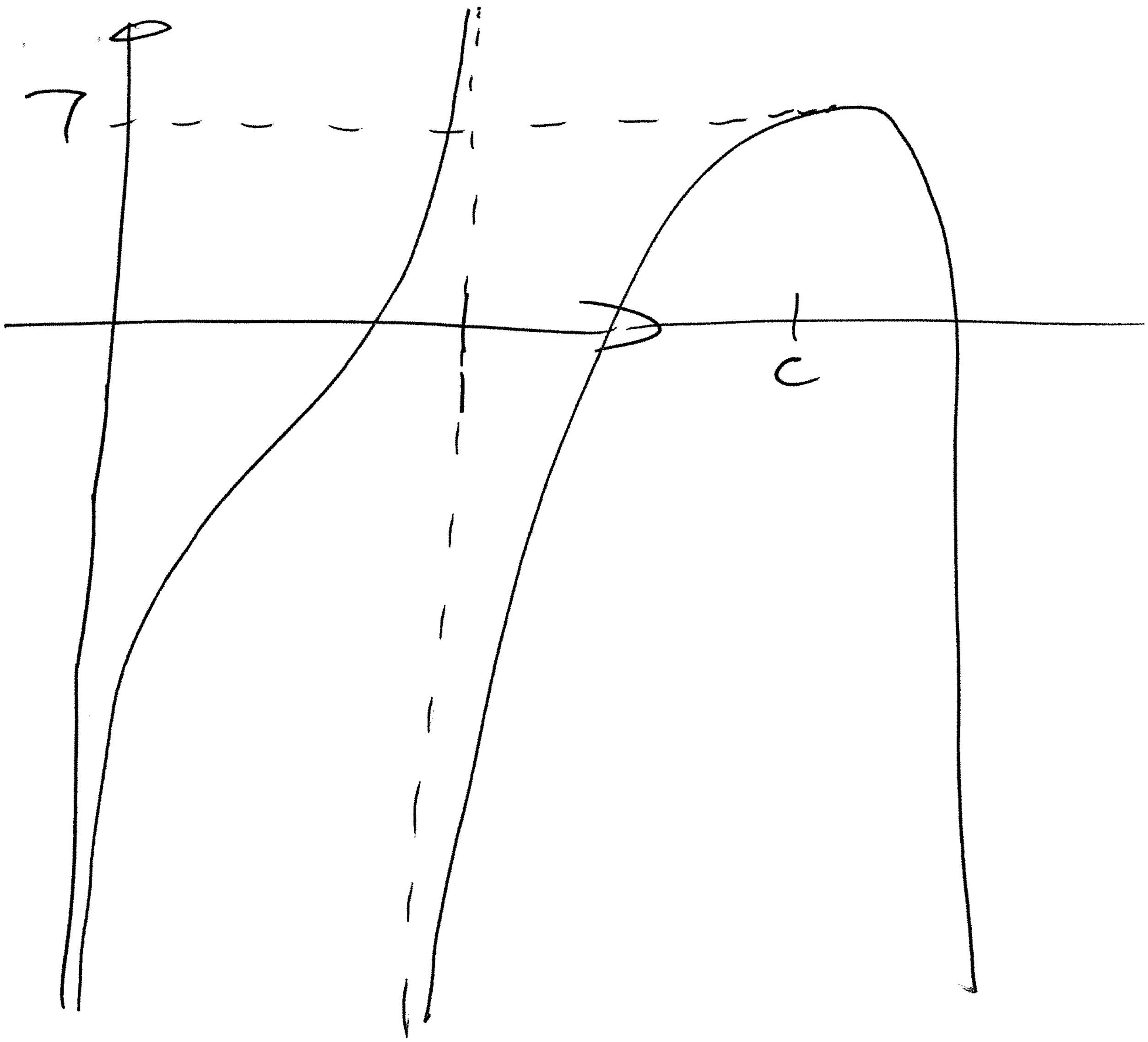
Note $f(x)$ Does **not** have an absolute max, at $x=5$. $x=5$ is **not** in the domain of f.



Alternate wording

absolute max = global max

absolute min = global min.



No absolute max or min.
Local max ~~at x=4~~ value
~~value~~ of 7 at $x=c$

Extreme Value Thm

If f is continuous on $[a,b]$ then f attains an absolute max & absolute min on $[a,b]$.

(Existence theorem; does not tell How to find them).

Extreme values: Extreme values of f are the ~~maximum &~~ absolute max & absolute min of f.

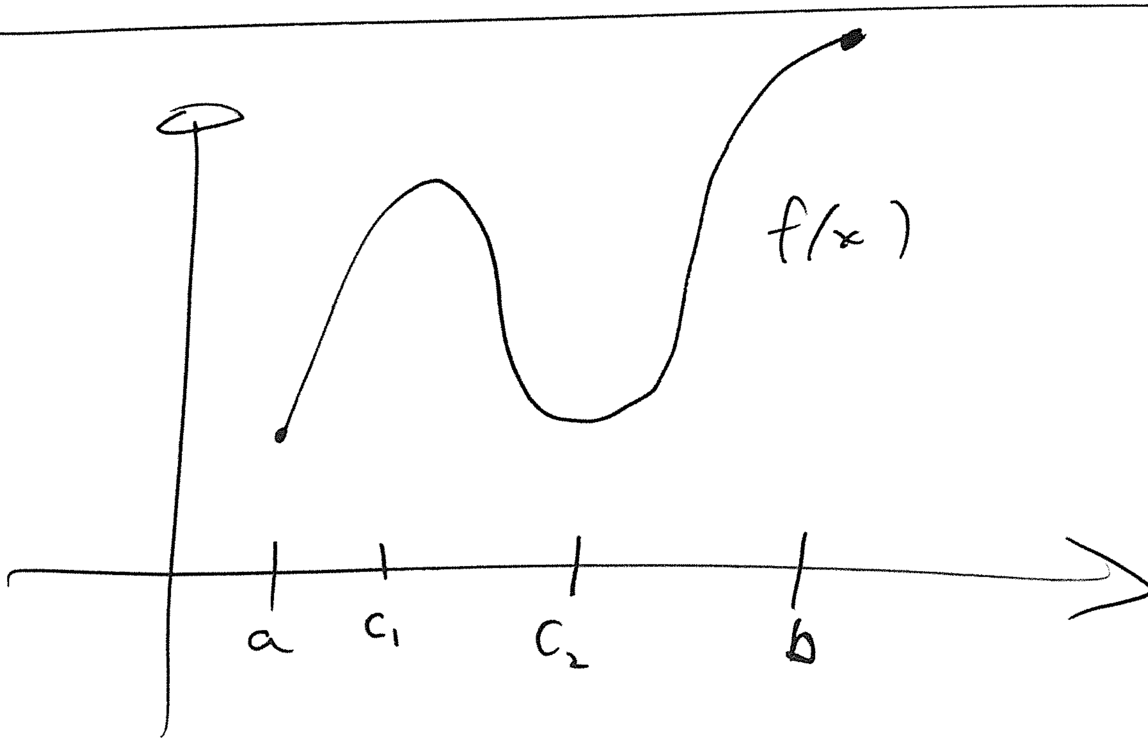
Finding c where f has ~~local~~ absolute or local extreme values.

~~Also note local Absolute~~
Also note global max/min are local max or min.

Fermat's thm

If f has a local max/min at c & c is <u>not</u> an endpoint of the domain, then

$f'(c) = 0$ or $f'(c)$ undefined.



f has local max at c_1 & $f'(c_1) = 0$

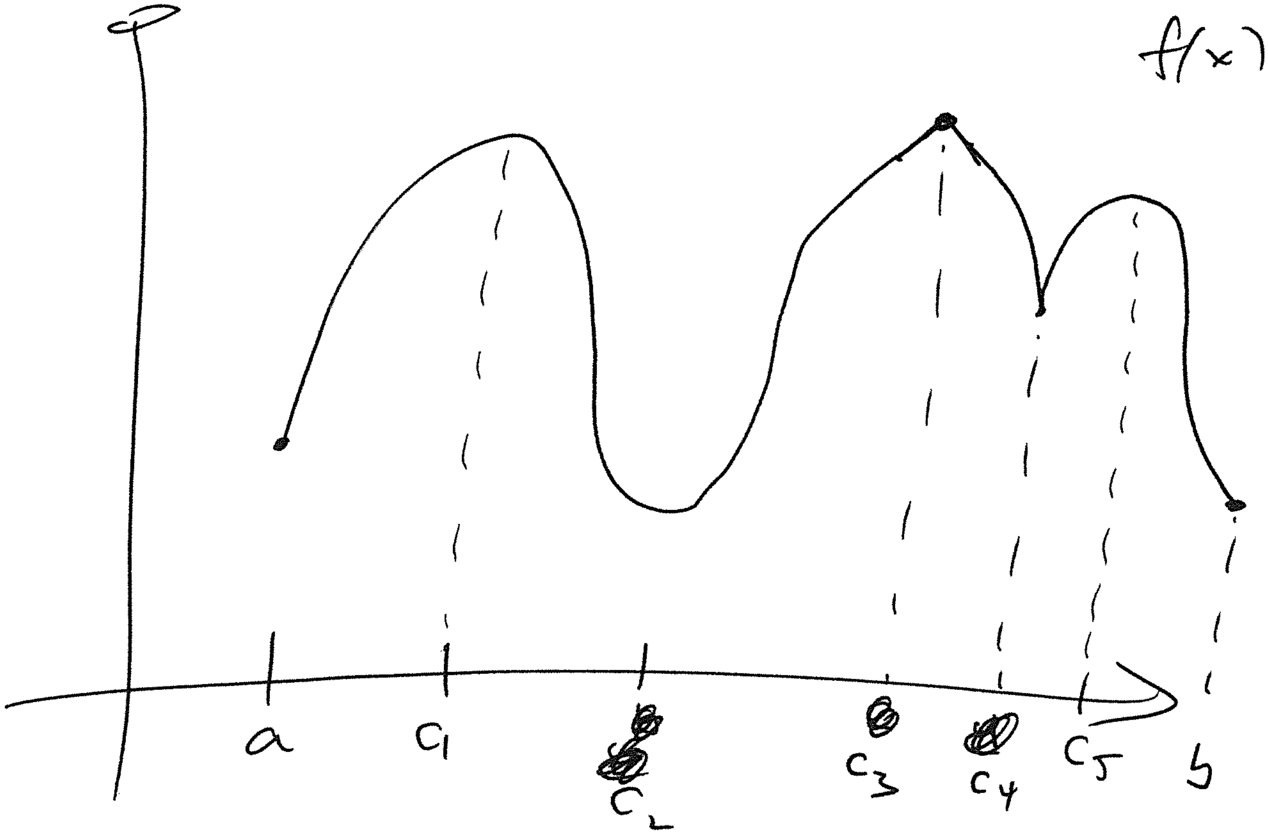
f has local min at c_2 & $f'(c_2) = 0$.

note f has a absolute min (hence also a local min) at $x=a$ but $f'(a) \neq 0$.

Similar at $x=b$.

Defn. A <u>critical number</u> (critical point) of a function f is a number c in the domain of f such that $f'(c)=0$ or $f'(c)$ does not <u>exist</u>; if the domain of f is an interval we Do <u>NOT</u> ~~include the ends~~ allow c to be an end point.

Ex Consider



the critical points of f are

c_1, c_2, c_3, c_4, c_5,

$f'(c_1) = f'(c_2) = f'(c_5) = 0$

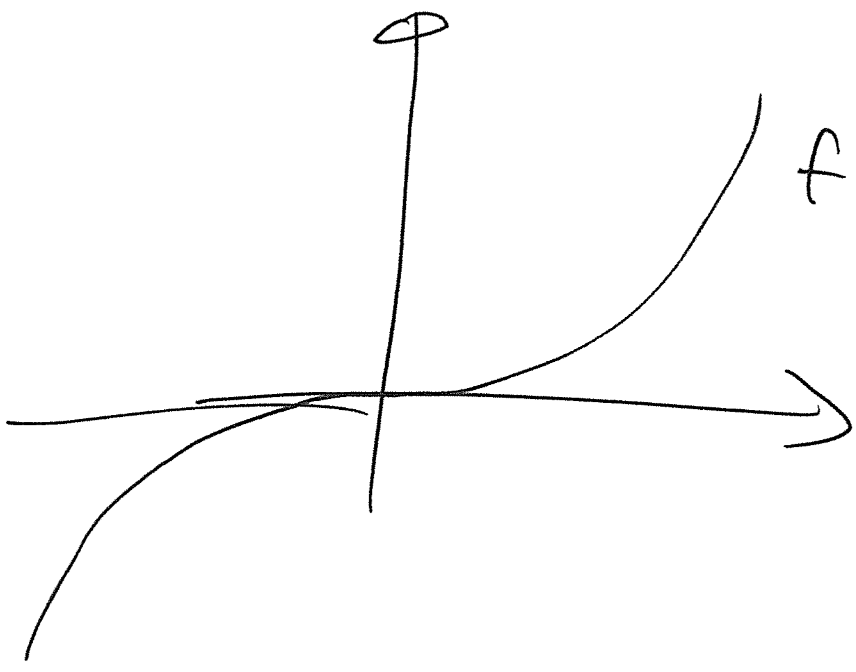
$f'(c_3)$ & $f'(c_4)$ not defined.

Even if $f'(a) = 0$ (a an endpoint)
or $f'(a)$ undefined

we don't allow it to be a critical point.

<u>Ex</u> $f(x) = x^3$. Then $f'(0) = 0$

but f <u>DOES NOT</u> have local max or local min at $x = 0$



the closed interval method.

To find an absolute max or min values of a continuous f on a closed interval $[a,b]$.

1. List all critical #'s c $c_1, c_2, \ldots \in (a,b)$ such that $f'(c)=0$ or $f'(c)$ undefined.

   Compute $f(c_1), f(c_2), \ldots,$

2. Compute $f(a)$ & $f(b)$.
3. The largest (smallest) values from step 1 & 2 is the absolute max value of (min).

f.

Ex 8 Find the absolute max & min values of

$f(x) = x^3 - 3x^2 + 1$ on $[-\frac{1}{2}, 4]$.

Solution,

solve $0 = f'(c)$ for $c \in (-\frac{1}{2}, 4)$.

$f'(x) = 3x^2 - 3(2)x$

$= 3x[x-2]$.

$f'(c) = 0 \quad c = 0$ & $c = 2$

Note no c such that $f'(c)$ Does not exist.

list $f(0) = 1$

$f(2) = -3$.

2) compute f at end points:

$f(-\frac{1}{2}) = \frac{1}{8}$, $f(4) = 17$.

3. $f(0) = 1$

$f(2) = -3$ — smallest.

$f(-\frac{1}{2}) = \frac{1}{8}$

$f(4) = 17$ — largest

f has a absolute max value of $f(4) = 17$,

f has an absolute min value of $f(2) = -3$.

<u>Ex</u> find the absolute max & min of $f(x) = x + \frac{1}{x}$ on $[-\frac{1}{2}, 3]$.

<u>Ex</u> Find the absolute max & min of $f(x) = |x| - x^2 - 3$ on $[-1, 4]$.

Found $f'(0)$ undefined

Start at $f'(c) = 0$.

4.2 Mean Value Theorem.

~~one~~ two main theorem.

~~Rolle's thm~~

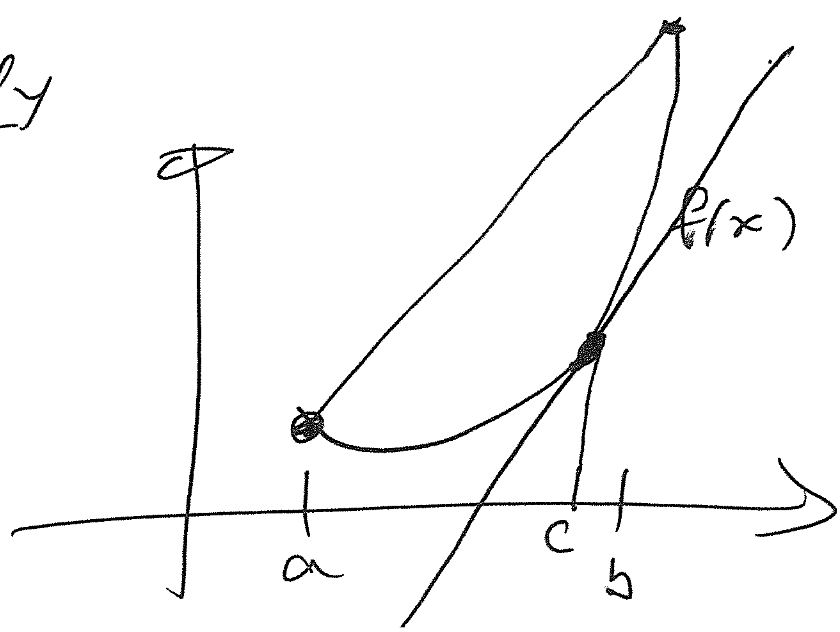
Mean Value Thm

Suppose f is differentiable on (a,b) & continuous on $[a,b]$.

then there is some $c \in (a,b)$ s.t.

$$f'(c) = \frac{f(b)-f(a)}{b-a},$$

graphically



c is some # in (a, b) ~~st~~.
such that the tangent line
to the curve at $(c, f(c))$ ~~to~~
is parrallel to the secant line
through $(a, f(a))$ & $(b, f(b))$.

Ex #11 pg 292

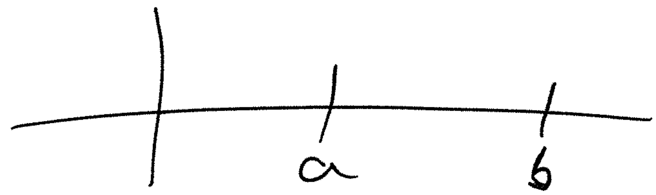
4.3 Derivatives & graphs.

Increasing/Decreasing test.

1. if $f'(x) > 0$ on an interval
then f increasing on the interval

2. if $f'(x) < 0$ on an interval
then f decreasing on the interval

Proof of ①



let $f'(x) > 0$ all $x \in (a, b)$.

let $x_1 < x_2$, $x_1, x_2 \in (a, b)$.

Goal (show $f(x_1) < f(x_2)$).

Since f differentiable on (x_1, x_2) &
continuous on $[x_1, x_2]$ we can
Apply MVT (mean value theorem)
to find $c \in (x_1, x_2)$ such that

$$f'(c) = \frac{f(x_2) - f(x_1)}{x_2 - (x_1)}$$

$$\Rightarrow f(x_2) - f(x_1) = \underbrace{f'(c)}_{>0} \underbrace{(x_2 - x_1)}_{>0} > 0$$

Ex Find where the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing & where it is decreasing?

Draw table on pg 294

function

The first derivative test

pg 294

by similar reasoning we see that $x=0$ is a <u>local max</u> and $x=3$ is a <u>local min</u>.

This method of finding local maxes and mins is called the <u>first derivative test</u>.

<u>First derivative test</u>:

Suppose c is a critical number of f.

(a) If $f'(x)$ is positive immediately to the left of c and negative immediately to the right, then $f(x)$ looks like $\nearrow \searrow$ so $f(c)$ is a local max.

(b) If $f'(x)$ is negative immediately to the left of c and positive immediately to the right, then $f(x)$ looks like $\searrow \nearrow$ and $f(c)$ is a local min.

(c) If $f'(x)$ doesn't change sign at c, then $f(c)$ is not a local max or local min.

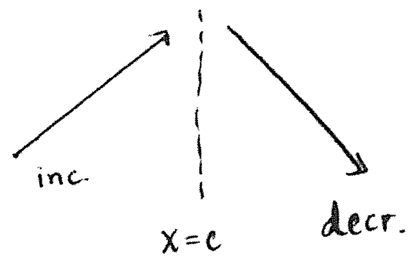
§4.3 continued.

Last day we saw that:

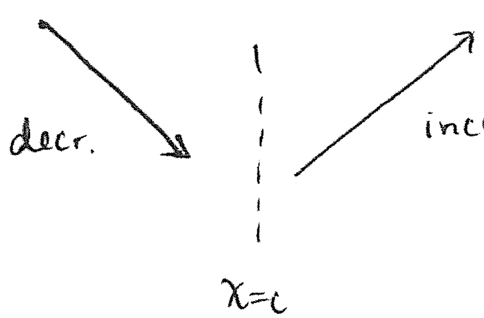
- If $f'(x) > 0$ on an interval (a,b) then $f(x)$ is increasing there
- If $f'(x) < 0$ on an interval (a,b) then $f(x)$ is decreasing there.

First derivative test:

Therefore, if $f'(x)$ changes from positive to negative at a critical value c, then $f(x)$ looks like:



So that $f(c)$ is a local max. Similarly if $f'(x)$ changes from negative to positive at a critical value c then $f(x)$ looks like



So $f(c)$ is a local minimum.

Example: If $f(x) = |x^2 - x|$, where are the local maxes and local mins?

Solution: We saw in a previous class that

$$f'(x) = \frac{x(x-1)(2x-1)}{|x^2-x|}$$ and the critical values are

$0, \frac{1}{2}, 1$. So we make a table (note that since $|x^2-x|$ is always positive we don't need it in the table:

function	$(-\infty, 0)$	$(0, \frac{1}{2})$	$(\frac{1}{2}, 1)$	$(1, \infty)$
x	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$
$2x-1$	$-$	$-$	$+$	$+$
$f'(x)$	$-$	$+$	$-$	$+$
$f(x)$	$\searrow$	$\nearrow$	$\searrow$	$\nearrow$

min max min

So we know that $f(0) = 0$ is a local min, so is $f(1) = 0$. On the other hand $f(\frac{1}{2}) = |(\frac{1}{2})^2 - \frac{1}{2}|$

$$= |\tfrac{1}{4} - \tfrac{1}{2}|$$

$$= \tfrac{1}{4}$$

is a local max.

—— END EXAMPLE ——

Now we know what the first derivative $f'(x)$ says about $f(x)$. What about $f''(x)$?

Definition: A graph is concave up if it looks like , i.e. all of its tangent lines are below the graph:

This happens exactly when $f''(x) > 0$. If $f''(x) < 0$, then the graph is concave down:

The points where $f''(x) = 0$ (i.e. $f(x)$ changes from concave down to convave up or vice versa) are called inflection points.

Example: Where is the function $f(x) = x^4 - 4x^3$ increasing, decreasing, concave up and concave down? What are the max/min and the inflection points?

Solution: The first derivative of f is

$$f'(x) = 4x^3 - 12x^2$$
$$= 4x^2(x-3)$$

So the critical values are $x=0$ and $x=3$.

So we get a table:

function	$(-\infty, 0)$	$(0, 3)$	$(3, \infty)$
$4x^2$	$+$	$+$	$+$
$x-3$	$-$	$-$	$+$
$f'(x)$	$-$	$-$	$+$
$f(x)$	$\searrow$ decr.	$\searrow$ decr	$\nearrow$ incr.

So $f(x)$ has a minimum at $f(3) = 3^4 - 4 \cdot 3^3 = -27$., is decreasing on $(-\infty, 3)$ and increasing on $(3, \infty)$.

Then
$$f''(x) = 12x^2 - 24x$$
$$= 12x(x-2)$$

So $f''(x) = 0$ gives $x=0$ and $x=2$. Then we get a table:

function	$(-\infty, 0)$	$(0, 2)$	$(2, \infty)$
$12x$	$-$	$+$	$+$
$(x-2)$	$-$	$-$	$+$
$f''(x)$	$+$	$-$	$+$
$f(x)$	$\cup$ conc. up.	$\cap$ conc. down	$\cup$ conc. up

The inflection points are at $x=0$ and $x=2$, they are $f(0) = 0$ and $f(2) = 2^4 - 4(2^3) = -16$. The graph is concave down on $(0,2)$, and concave up on $(-\infty, 0)$ and $(2, \infty)$.

Note we can also see from the second derivative that $f(3)$ is a local min, because $f(x)$ is concave up there.

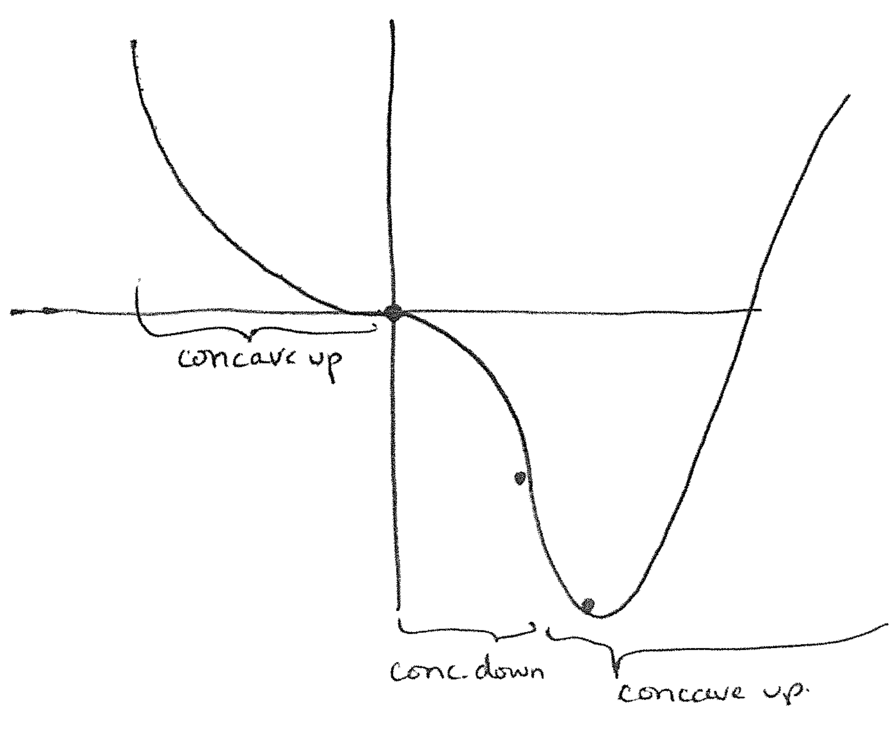
Second derivative test: Suppose f'' is cts near c.

(i) If $f'(c)=0$ and $f''(c)>0$, then $f(x)$ has a local minimum at c ($f(x)$ looks like )

(ii) If $f'(c)=0$ and $f''(c)<0$, then $f(x)$ has a local maximum at c ($f(x)$ looks like )

So we can sketch $f(x)$ by first:

- Plot the maxes/mins (min @ $(3,-27)$)
- Add the inflection points $(0,0)$ and $(2,-16)$.



Note that this curve is still somewhat "sketchy".
We will spend 2 more classes refining our sketching techniques.

Also: Notice that the second derivative test can fail!
If $f''(c)=0$, as happened in the previous example with $c=0$, we learn no information about whether or not c is a local max or local min.