Section 3.4 (The chain rule) L2

If $f(x)$ & $g(x)$ are differentiable functions then

$$(f\circ g)'(x) = [f(g(x))]' = f'(g(x))\, g'(x).$$

In the Leibniz notation if $y = f(u)$ & $u = g(x)$ then

$$\frac{dy}{dx} = \frac{df}{du}\,\frac{du}{dx}$$

Note the "du"'s cancel & we get $\frac{dy}{dx} = \frac{df}{dx}$

<u>notation</u> this notation is better for applications & related rates

Ex let $F(x) = \sqrt{x^2+1}$

Find $F'(x)$.

solution. Need to write

$F(x) = f[g(x)]$ & then Apply

chain rule.

inside function

outside function

$F(x) = f[g(x)]$

$g(x) = x^2+1$

$f(x) = \sqrt{x}$

$g'(x) = 2x$

$f'(x) = \dfrac{1}{2\sqrt{x}}$

So by the chain rule

$$F'(x) = f'(g(x))\ g'(x).$$

$$= f'(g(x))\ (2x)$$

$$= \frac{1}{2\sqrt{g(x)}}\ (2x)$$

$$= \frac{1}{2\sqrt{x^2+1}}\ (2x)$$

<u>Note</u> Its $f'(g(x))$ <u>not</u> $f'(x)$.

<u>Ex</u> Compute $F'(x)$ if
$F(x) = \sin(x^2)$.

<u>solution</u>

write $F(x) = f[g(x)]$

$g(x) = x^2$ $\quad f(x) = \sin(x)$

$g'(x) = 2x$ $\quad f'(x) = \cos(x)$

$F'(x) = f'(g(x)) \; g'(x)$

$= \cos(g(x)) \; (2x)$

$= \cos[x^2] \; (2x)$

After about 5 examples you won't explicitly write out $F(x) = f[g(x)]$.

Alternate notation (Leibniz).

let $y = \frac{1}{\sqrt{2x-1}}$. Compute $\frac{dy}{dx}$.

Solution.

$y = (2x-1)^{-\frac{1}{2}}$

$y = f(u) = u^{-\frac{1}{2}}$ $\qquad u = u(x) = 2x-1$.

then

$\frac{dy}{dx} = \frac{df}{du}\frac{du}{dx}$ $\qquad$ (but write final Answer in just x)

$\frac{df}{du} = -\frac{1}{2}u^{-\frac{3}{2}}$, $\qquad \frac{du}{dx} = 2$.

So

$$\frac{dy}{dx} = \left(-\frac{1}{2}u^{-\frac{3}{2}}\right)(2)$$

$$= -\frac{1}{2}\left[2x-1\right]^{-\frac{3}{2}}(2).$$

write u in terms of x

(don't want any u's in final answer)

3

6ii

<u>Ex</u> Compute

$$\frac{d}{dx} e^{[\cos(x)+x]}.$$

<u>Solution</u> outside function is $e^{\square}$; inside is $\cos(x)+x$

So $$\frac{d}{dx} e^{[\cos(x)+x]} = e^{(\cos(x)+x)}\left(-\sin(x)+1\right).$$

<u>Ex</u> If n is a real number then

$$\frac{d}{dx}(g(x))^n = n(g(x))^{n-1} g'(x).$$

<u>Proof</u>

$$F(x) = (g(x))^n = f[g(x)] \qquad f(x) = x^n$$

$$f'(x) = nx^{n-1}$$

so

$$F'(x) = f'(g(x))\, g'(x)$$
$$= n\,(g(x))^{n-1}\, g'(x).$$

<u>Ex</u> Compute $h'(t)$ if

$$h(t) = \left(\frac{t-2}{2t+1}\right)^9.$$

$$h(t) = f[g(t)] \qquad g(t) = \frac{t-2}{2t+1}$$

$$h'(t) = f'(g(t))\, g'(t) \qquad f(t) = t^9$$

$$= 9(g(t))^8\, g'(t) \qquad f'(t) = 9t^8$$

$$= 9\left[\frac{t-2}{2t+1}\right]^8 g'(t) \qquad g'(t) = \frac{(1(2t+1) - 2(t-2))}{(2t+1)^2}$$

$$= 9\left[\frac{t-2}{2t+1}\right]^8 \left[\frac{(2t+1)-2(t-2)}{(2t+1)^2}\right]$$

Ex Using chain rule & product rule to prove Quotient rule.

Sol.

Recall $\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$.

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{d}{dx}\left(f\left(g^{-1}\right)\right)$$

$$= f'(x)\frac{1}{g(x)} + f(x)\frac{d}{dx}\left(\left(g(x)\right)^{-1}\right)$$

$$\frac{d}{dx}\left[\underbrace{(\underbrace{g(x)}_{\text{inner}})^{-1}}_{\text{outter}}\right] = (-1)\left[g(x)\right]^{-1-1} g'(x)$$

$$= -\frac{g'(x)}{(g(x))^2}$$

So

$$\frac{d}{dx}\left(\frac{f}{g}\right) = \frac{f'(x)}{g(x)} + f(x)\left[-\frac{g'(x)}{g(x)^2}\right]$$

$$= \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}.$$

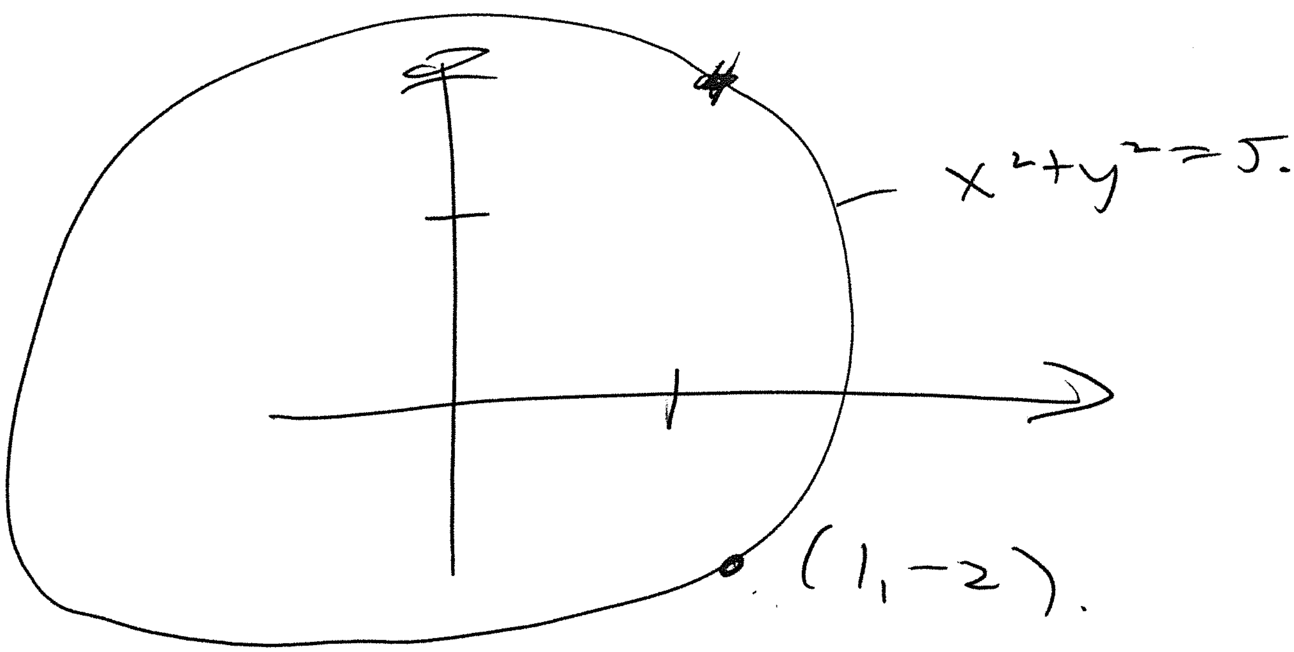
Section 3.5 (Implicit Differentiation).

We use implicit differentiation to compute derivatives of a curve that is written in implicit form.

Ex Consider the curve

$x^2+y^2=5$.

Compute the slope of the tangent line at the point $(x,y) = (1,-2)$



Method 1 (cannot always do). L11

Solve for y.

$$y^2 = 5 - x^2$$

$$y = -\sqrt{5-x^2}.$$

Then $$\left.\frac{dy}{dx}\right|_{x=1} = \left.-\frac{(-2x)}{2(5-x^2)^{\frac{1}{2}}}\right|_{x=1}$$

$$= \left.\frac{2x}{2(5-x^2)^{\frac{1}{2}}}\right|_{x=1}$$

$$= \frac{2}{2\sqrt{4}} = \frac{1}{2}.$$

Method 2 (Implicit derivative).

consider $y = y(x)$ so

we have $x^2 + [y(x)]^2 = 5$ all x.

So we can take a derivative with respect $\frac{d}{dx}$ of both sides.

$$0 = \frac{d}{dx}(5) = \frac{d}{dx}\left[x^2 + [y(x)]^2\right]$$

$$= 2x + \frac{d}{dx}[y(x)]^2$$

$$= 2x + 2y(x)\,y'(x).$$

So $\frac{dy}{dx} = -\frac{x}{y}$.

So $\left.\frac{dy}{dx}\right|_{(x,y)=(1,-2)} = \frac{-1}{-2} = \frac{1}{2}$.

Ex Suppose A curve is given by

$x^2 + xy + \sin(y) = 1.$

$+ y^3$

Show the point $(x, y) = (1, 0)$

lies on the curve.

Compute $\frac{dy}{dx}$ at this point.

Solution

$1^2 + 1 \cdot 0 + \sin(0) = 1$ so $(1, 0)$

$+ 0^3$

lies on the curve.

Take derivatives of both sides with respect to x

(& Assume $y = y(x)$).

$$0 = \frac{d}{dx}(1) = \frac{d}{dx}\left[x^2 + xy + \sin(y) + y^3\right]$$

$$= 2x + \frac{d}{dx}(xy) + \frac{d}{dx}\sin(y) + \frac{d}{dx}(y^3)$$

$$= 2x + \left[y + x\frac{dy}{dx}\right] + \cos(y)\frac{dy}{dx} + 3y^2\frac{dy}{dx}.$$

To compute $\left.\frac{dy}{dx}\right|_{(1,0)}$ sub in values of x & y.

15

$$0 = 2(1) + \left[0 + x\frac{dy}{dx}\right]$$

$$+ \cos(0)\frac{dy}{dx} + \cancelto{0}{3(0^2)}\frac{dy}{dx}$$

$$0 = 2 + \frac{dy}{dx} + \frac{dy}{dx}$$

so $\frac{dy}{dx} = -1$

$$\boxed{\left.\frac{dy}{dx}\right|_{(1,0)} = -1}$$

Ex Suppose $x^3 + xy + y^2 = \cancel{} x^4 + y$

Compute $\frac{dy}{dx}$ (in your Answer you will have an x & y).

~~[illegible]~~

Solution.

~~Ex show the point (1,0) lies on the curve~~

$\cancel{xy^3 + xy^2 = \sin(y).}$

~~Find the tangent line to the curve at this point.~~

Ex Find $\frac{dy}{dx}$ if $e^{(y^2)} = xy$.

solution

$$\frac{d}{dx}(xy) = \frac{d}{dx}\left(e^{y^2}\right)$$

$$y + xy' = e^{y^2}\,2yy'$$

~~So~~ now isolate y' (can Always do this).

18

$$y = y'(2ye^{y^2} - x],$$

$$y' = \frac{y}{2ye^{y^2} - x}.$$

$$\frac{dy}{dx} = \frac{y}{2ye^{y^2} - x}.$$

Related rates | Section 3.9

Related rates is a real world Application.

① you are given the rate of change of some quantity (ie $\frac{d}{dt}$ of the quantity) &

Asked something about the rate of change of Another quantity.

Ex Air is being pumped into a spherical balloon so its volume increases at a rate of $100\frac{cm^3}{s}$.

How fast is the radius of the balloon increasing when the ~~radius~~ diameter is 50 cm.

<u>Solution</u> We identify & name the the changing quantities.

<u>Quantity 1</u> Volume of balloon is $V(t) = V$.

<u>Quantity 2</u> Radius of balloon say $r = r(t)$.

<u>Given Quantity</u>

$\frac{dV}{dt}$ = rate of change of volume $= 100 \frac{cm^3}{sec}$

<u>Want to find</u>: $\frac{dr}{dt}$ when the diameter is 50 cm (i.e. when $r = 25$).

NEED to find equation relating V & r.

$$V = \frac{4}{3}\pi r^3.$$

Now recall we are thinking of each quantity as a function of t;

$V(t) = \frac{4}{3}\pi (r(t))^3$. Now take a derivative in t;

So $V'(t) = 4\pi (r(t))^2 \, r'(t)$.

In the Leibniz notation (which is generally better for related rates)

(22)

$$\frac{dv}{dt} = \frac{dv}{dr}\frac{dr}{dt}$$

(As a check you can "cancel" the dr's on the right)

we now plug in all the known quantities.

$\frac{dv}{dr}$ $\quad \frac{dv}{dt} = 100$ & isolate

want we want (ie $\frac{dr}{dt}$).

$$100 = (4\pi r^2)\frac{dr}{dt}$$

but we want $\frac{dr}{dt}$ when $r = 25$

$$100 = 4\pi (25)^2 \left.\frac{dr}{dt}\right|_{r=25}$$

so $\left.\frac{dr}{dt}\right|_{r=25} = \frac{100}{4\pi(25)^2} \frac{cm}{sec}$

make sure to Add correct units.

For related rates question they will probably Ask ~~to~~ you to "State your final Answer in a complete sentence".

When the diameter is 50 cm the radious of the balloon is increasing at a rate of

$\frac{100}{4\pi(25)^2} \frac{cm}{sec}$.

23i

In previous example
we didn't need A picture.
In most related rates
problems you will need picture

<u>Important</u> For ANY quantities
that are changing; do <u>NOT</u> sub
in ANY values till After
taking derivative. ~~[illegible]~~

Ex 2 pg 246

A ladder 10 ft long rests against a vertical wall....

~~Ex 3 pg 246~~

Start Here

Ex 3 pg 246.