

A note on the nonexistence of positive supersolutions to elliptic equations with gradient terms

A. Aghajani* C. Cowan†

April 10, 2020

Abstract

We prove that if the elliptic problem $-\Delta u + b(x)|\nabla u| = c(x)u$ with $c \geq 0$ has a positive supersolution in a domain Ω of $\mathbb{R}^{N \geq 3}$, then c, b must satisfy the inequality

$$\sqrt{\int_{\Omega} c\phi^2} \leq \sqrt{\int_{\Omega} |\nabla\phi|^2} + \sqrt{\int_{\Omega} \frac{b^2}{4}\phi^2}, \quad \phi \in C_c^{\infty}(\Omega).$$

As an application, we obtain Liouville type theorems for positive supersolutions in exterior domains when $c(x) - \frac{b^2(x)}{4} > 0$ for large $|x|$, but unlike the known results we allow the case $\liminf_{|x| \rightarrow \infty} c(x) - \frac{b^2(x)}{4} = 0$. Also the weights b and c are allowed to be unbounded. In particular, among other things, we show that if $\tau := \limsup_{|x| \rightarrow \infty} |xb(x)| < \infty$ then this problem does not admit any positive supersolution if

$$\liminf_{|x| \rightarrow \infty} |x|^2 c(x) > \frac{(N-2+\tau)^2}{4},$$

and, when $\tau = \infty$, we have the same if

$$\limsup_{R \rightarrow \infty} R \left(\frac{\inf_{R < |x| < 2R} (c(x) - \frac{b(x)^2}{4})}{\sup_{\frac{R}{2} < |x| < 4R} |b(x)|} \right) = \infty.$$

2010 Mathematics Subject Classification Primary 35J60; Secondary 35B53..

Key words: Liouville type theorems; supersolutions, gradient term.

*School of Mathematics, Iran University of Science and Technology, Narmak, Tehran, Iran. Email: aghajani@iust.ac.ir.

†Department of Mathematics, University of Manitoba, Winnipeg, Manitoba, Canada R3T 2N2. Email: craig.cowan@umanitoba.ca. Research supported in part by NSERC.

1 Introduction and statement of the main results

In this work we consider classical supersolutions of the equation

$$-\Delta u + b(x)|\nabla u| = c(x)u, \quad x \in \Omega, \quad (1)$$

where Ω is an exterior domain \mathbb{R}^N , $N \geq 3$. By a classical supersolution we mean a function $u \in C^2(\Omega)$ verifying the inequality $-\Delta u + b(x)|\nabla u| \geq c(x)u$ pointwise in Ω .

In [8] (also see [9]), Berestycki, Hamel and Nadirashvili implicitly proved, as a consequence of the study of eigenvalue problems in \mathbb{R}^N , that if b, c are continuous functions in \mathbb{R}^N then the problem

$$-\Delta u + b(x) \cdot \nabla u \geq c(x)u \quad \text{in } \mathbb{R}^N \quad (2)$$

does not admit any positive solution provided that b and c are bounded and satisfy

$$\liminf_{|x| \rightarrow \infty} c(x) - \frac{b(x)^2}{4} > 0. \quad (3)$$

L. Rossi in [22] generalized the above nonexistence results to the framework of fully nonlinear elliptic operators. As a particular case, it follows that if b, c are bounded in $\mathbb{R}^N \setminus B_{R_0}$ and (3) holds then Problem 2 does not admit any positive solution. It's worth noting that every supersolution u of (2) is also a supersolution of (1) as we have $b(x) \cdot \nabla u \leq |b(x)||\nabla u|$. To see some related problems one can see [11] and [13] where the authors proved some Hadamard and Liouville type properties for nonnegative viscosity supersolutions of fully non linear uniformly elliptic partial differential inequalities in the whole space, or in an exterior domain, for more references see [3, 4, 5, 6, 7, 10, 11, 13, 14, 15, 18].

In [3] Alarcon, Garcia-Melian and Quaas considered positive classical supersolutions of (1) for more general unbounded weights b and c . They proved that if $b, c \in C(\mathbb{R}^N \setminus B_{R_0})$ verify (3) and satisfy a further restriction related to the fundamental solutions of the homogeneous problem (see Theorems 1.1 and 1.2 in [3]) then there are no classical positive supersolutions to (1) which do (or do not) blow up at infinity. Their proof of nonexistence results depends on properties of the function $m(R) = \inf_{|x|=R} u(x)$ and fundamental solutions of the equation $\Delta v + \tilde{b}(|x|)|\nabla v| = 0$ in $\mathbb{R}^N \setminus B_{R_0}$, where $\tilde{b}(r) := \sup_{|x|=r} b(x)$.

In this paper we use a different approach, by employing a generalized version of Hardy inequality, and obtain new Liouville type results, that seems to be sharp in some sense, and improve the results mentioned above. In particular, we may allow the case

$$\liminf_{|x| \rightarrow \infty} c(x) - \frac{b^2(x)}{4} = 0,$$

and without the boundedness assumption on the weights b and c .

We proceed now to the statement of our main results. First, using a generalized version of Hardy inequality proved by the second author in [16], we prove the following lemma which is crucial in the proofs of the main results.

Proposition 1. *If (1) has a solution $u > 0$ then we have*

$$\sqrt{\int_{\Omega} c\phi^2} \leq \sqrt{\int_{\Omega} |\nabla\phi|^2} + \sqrt{\int_{\Omega} \frac{b^2}{4}\phi^2}, \quad (4)$$

or equivalently,

$$\int_{\Omega} (c - \frac{b^2}{4})\phi^2 \leq \int_{\Omega} |\nabla\phi|^2 + 2\sqrt{\int_{\Omega} |\nabla\phi|^2 \int_{\Omega} \frac{b^2}{4}\phi^2} \quad (5)$$

for every $\phi \in C_c^{\infty}(\Omega)$.

The following is our first general nonexistence result.

Theorem 1. *Let $\Omega = \mathbb{R}^N \setminus B_{R_0}$ be an exterior domain and $b, c \in C(\Omega)$ with $c(x) - \frac{b^2(x)}{4} > 0$ for $|x|$ sufficiently large. Then (1) does not have any positive supersolution if for some $\gamma > 1$ we have*

$$\sup_{R > 2R_0} \frac{\inf_{R < |x| < \gamma R} |x|^2(c - \frac{b^2}{4})}{\sup_{\frac{R}{2} < |x| < 2\gamma R} (1 + \frac{2|xb(x)|}{N-2})} > \frac{\beta^2 \ln 2 + 4\beta + 6}{\ln \gamma} + \beta^2, \quad (6)$$

where $\beta := \frac{N-2}{2}$. In particular this the case if

$$\frac{\liminf_{|x| \rightarrow \infty} |x|^2(c - \frac{b^2}{4})}{\limsup_{|x| \rightarrow \infty} (1 + \frac{2|xb(x)|}{N-2})} > \frac{(N-2)^2}{4}, \quad (7)$$

As a consequence we have the following result which is more easy to be checked in applications:

Proposition 2. *Let $b, c \in C(\mathbb{R}^N \setminus B_{R_0})$ with $c(x) - \frac{b^2(x)}{4} > 0$ for $|x|$ sufficiently large.*

(i) *Assume $\tau := \limsup_{|x| \rightarrow \infty} |xb(x)| < \infty$. Then (1) does not have any positive supersolution if*

$$\liminf_{|x| \rightarrow \infty} |x|^2 c(x) > \frac{(N-2+\tau)^2}{4}. \quad (8)$$

Moreover, if $b \geq 0$ and

$$\limsup_{|x| \rightarrow \infty} |x|^2 c(x) < \frac{(N-2+\tau)^2}{4}, \quad (9)$$

then (1) has a positive supersolution in $\mathbb{R}^N \setminus B_{R_1}$ for R_1 sufficiently large.

(ii) *If $\limsup_{|x| \rightarrow \infty} |xb(x)| = \infty$ then (1) does not have any positive supersolution if*

$$\limsup_{R \rightarrow \infty} R \left(\frac{\inf_{R < |x| < 2R} (c - \frac{b^2}{4})}{\sup_{\frac{R}{2} < |x| < 4R} |b(x)|} \right) = \infty. \quad (10)$$

Example 1. Consider the problem

$$-\Delta u + b|x|^\lambda |\nabla u| \geq c|x|^\mu u \quad \mathbb{R}^N \setminus B_{R_0}, \quad (11)$$

where $b, c \in \mathbb{R}$ and $\mu \geq 2\lambda$. Then it is easy to see that we have

$$J(R) := R \left(\frac{\inf_{R < |x| < 2R} (c(x) - \frac{b(x)^2}{4})}{\sup_{R < |x| < 2R} |b(x)|} \right) \geq C_0 R^{\mu-\lambda}, \quad \text{when } \mu > 2\lambda,$$

and also

$$J(R) = C_1 \left(c - \frac{b^2}{4} \right) R^\lambda, \quad \text{when } \mu = 2\lambda.$$

Then by Corollary 2 we see that (11) does not admit any positive supersolution when $\mu > 2\lambda$ and $c > 0$, or $\mu = 2\lambda$ and $c - \frac{b^2}{4} > 0$. Also, in the remaining cases, its not hard to see that a positive supersolution can always be constructed for suitably large R_0 (see [3] Corollary 1.2).

Example 2. Consider the problem

$$-\Delta u + |x|^\lambda |\nabla u| \geq \left(\frac{|x|^{2\lambda}}{4} + \frac{1}{|x|^\mu} \right) u, \quad x \in \mathbb{R}^N \setminus B_1, \quad (12)$$

where $\lambda > -1$ and $\lambda + \mu < 1$. Note here we have $c(x) - \frac{b(x)^2}{4} = \frac{1}{|x|^\mu}$, hence, if $\mu > 0$

$$\liminf_{|x| \rightarrow \infty} c - \frac{b^2}{4} = 0,$$

thus none of the previous results can apply. However we have, for a fixed $\gamma > 1$,

$$\frac{\inf_{R < |x| < \gamma R} |x|^2 (c - \frac{b^2}{4})}{1 + \frac{2 \sup_{R < |x| < \gamma R} |x b(x)|}{N-2}} = \frac{\inf_{R < |x| < \gamma R} |x|^{2-\mu}}{1 + \frac{2 \sup_{R < |x| < \gamma R} |x|^{1+\lambda}}{N-2}} = O(R^{1-\mu-\lambda}) \rightarrow \infty \quad \text{as } R \rightarrow \infty.$$

Hence by the above result (12) does not admit any positive supersolution.

Remark 1. Note in the special case $b(x) \equiv 0$, from Proposition 2 (with $\tau = 0$ in part (i)) we see that the equation

$$-\Delta u \geq c(x)u, \quad (13)$$

in an exterior domain Ω , does not admit any positive supersolution if

$$\liminf_{|x| \rightarrow \infty} |x|^2 c(x) > \frac{(N-2)^2}{4}. \quad (14)$$

It is worth noting that in [17, 20, 21] it is pointed out that the above nonexistence result for positive supersolution to (13) can be obtained by using Agmon-Allegretto-Piepenbrink theory [1]. The above is also proved in [14] by a different method, where the authors for the

proof employed the Kelvin transform to transfer the unbounded domain Ω into a bounded one containing the origin and then applied a result of [15] regarding the nonexistence of positive solutions for the problem $-\Delta u = \frac{\mu}{|x|^2}u + f$, with Dirichlet BC. in a a bounded smooth domain D containing the origin, where $0 \leq f \in L_{loc}^\infty(D \setminus \{0\})$ and $\mu > \frac{(N-2)^2}{4}$.

However, note that our Proposition 2 proves more, indeed by part (i) we see that (14) is indeed sufficient for the nonexistence of positive supersolutions for the more general equation

$$-\Delta u + \frac{\varepsilon(x)}{|x|}|\nabla u| \geq c(x)u, \quad (15)$$

in exterior domains when $\lim_{x \rightarrow \infty} |\varepsilon(x)| = 0$.

As a byproduct of the above results, we can prove the following useful general estimate using a Hardy type inequality.

Corollary 1. *Let E be a positive smooth function in an exterior domain Ω in \mathbb{R}^N ($N \geq 3$) with $-\Delta E \geq 0$. Then*

$$\liminf_{|x| \rightarrow \infty} |x|^2 \frac{-\Delta E}{E} \leq \frac{(N-2)^2}{4}. \quad (16)$$

Example 3. *As an application of the above corollary consider the equation*

$$-\Delta u = |x|^a u^p, \quad \text{in } \Omega, \quad (17)$$

where $a \in \mathbb{R}$, $p > 1$ and Ω is an exterior domain in \mathbb{R}^N ($N \geq 3$). Now if u is a positive classical supersolutions of this equation then we get, by Corollary 1,

$$\liminf_{|x| \rightarrow \infty} |x|^2 \frac{-\Delta u}{u} = \liminf_{|x| \rightarrow \infty} |x|^{a+2} u^{p-1} \leq \frac{(N-2)^2}{4}.$$

However, we know that a superharmonic function u satisfies $u(x) \geq C|x|^{2-N}$ in Ω (see [23] or [2, 12]), hence we must have $a + 2 + (p-1)(2-N) \leq 0$. Thus the above equation does not admit any positive supersolution if $p < \frac{N+a}{N-2}$, which is a known result. Also, by a similar argument from Corollary 1 we see that the equation

$$-\Delta u = \frac{\mu u}{|x|^2}, \quad (\mu > 0)$$

does not admit any positive supersolution in an exterior domain if $\mu > \frac{(N-2)^2}{4}$.

Corollary 2. *If (1) has a solution $u > 0$, and there exists a smooth function $E > 0$ with $-\Delta E \geq 0$ such that*

$$b^2 \leq \gamma^2 \frac{|\nabla E|^2}{E^2} \quad (18)$$

then

$$\int_{\Omega} \left(c - \frac{b^2}{4}\right) \phi^2 \leq (1 + 2\gamma) \int_{\Omega} |\nabla \phi|^2, \quad \phi \in C_c^{\infty}(\Omega). \quad (19)$$

As a consequence (1) does not have any supersolution if

$$\liminf_{|x| \rightarrow \infty} |x|^2 \left(c - \frac{b^2}{4}\right) > (1 + 2\gamma) \frac{(N - 2)^2}{4}. \quad (20)$$

In particular, taking $E(x) = |x|^{2-N}$ we see that if $\tau := \limsup_{|x| \rightarrow \infty} |x|b(x) < \infty$ then (1) does not have any supersolution if

$$\liminf_{|x| \rightarrow \infty} |x|^2 c(x) > \frac{(N - 2 + \tau)^2}{4}.$$

2 Proofs of the main results

For the proof of our main results we use the following Hardy type inequality which is a special case of a result from [16]. For the sake of completeness we give a proof.

Lemma 1. *Let $E > 0$ be smooth. Then for all $T \in \mathbb{R}$ we have*

$$\int_{\Omega} |\nabla \phi|^2 dx \geq (T - T^2) \int \frac{|\nabla E|^2}{E^2} \phi^2 + T \int \frac{-\Delta E}{E} \phi^2 \quad \phi \in C_c^{\infty}(\Omega). \quad (21)$$

In particular, taking $T = \frac{1}{2}$ we get

$$2 \int_{\Omega} |\nabla \phi|^2 dx \geq \frac{1}{2} \int \frac{|\nabla E|^2}{E^2} \phi^2 + \int \frac{-\Delta E}{E} \phi^2 \quad \phi \in C_c^{\infty}(\Omega). \quad (22)$$

Proof. Fix $\phi \in C_c^{\infty}(\Omega)$ and set $v := E^{-T} \phi$. Then computing $E^{2T} |\nabla v|^2$ gives

$$E^{2T} |\nabla v|^2 = |\nabla \phi|^2 - 2T \nabla \phi \cdot \nabla E \phi E^{-1} + \frac{T^2 |\nabla E|^2 \phi^2}{E^2},$$

and now integrate this and note this term on the left is nonnegative. Now integrating the middle term by parts (put all derivatives on E) then gives the desired result. \square

Proof of proposition 1. For $t > \frac{1}{2}$ set $v := u^{\frac{1}{t}}$. Then we have

$$\frac{-\Delta v}{v} \geq \frac{c}{t} + (t - 1) \frac{|\nabla v|^2}{v^2} - b \frac{|\nabla v|}{v} \quad \text{in } \Omega,$$

or,

$$\begin{aligned} \frac{-\Delta v}{v} + \frac{1}{2} \frac{|\nabla v|^2}{v^2} &\geq \frac{c}{t} + (t - \frac{1}{2}) \frac{|\nabla v|^2}{v^2} - b \frac{|\nabla v|}{v} \\ &\geq \frac{c}{t} - \frac{b^2}{2(2t - 1)} \quad \text{in } \Omega. \end{aligned}$$

Recall the inequality (22),

$$\int_{\Omega} \frac{-\Delta v}{v} \phi^2 dx + \frac{1}{2} \int_{\Omega} \frac{|\nabla v|^2}{v^2} \phi^2 dx \leq 2 \int_{\Omega} |\nabla \phi|^2 dx$$

for all $\phi \in C_c^\infty(\Omega)$, then we get from above

$$\int_{\Omega} c \phi^2 dx \leq 2t \int_{\Omega} |\nabla \phi|^2 dx + \frac{t}{2(2t-1)} \int_{\Omega} b^2 \phi^2 dx.$$

Now we set

$$t = \frac{1}{2} + \frac{\sqrt{\int_{\Omega} b^2 \phi^2 dx}}{4\sqrt{\int_{\Omega} |\nabla \phi|^2 dx}}$$

to get

$$\int_{\Omega} c \phi^2 dx \leq \left(\sqrt{\int_{\Omega} |\nabla \phi|^2 dx} + \sqrt{\int_{\Omega} \frac{b^2}{4} \phi^2 dx} \right)^2$$

or

$$\sqrt{\int_{\Omega} c \phi^2 dx} \leq \sqrt{\int_{\Omega} |\nabla \phi|^2 dx} + \sqrt{\int_{\Omega} \frac{b^2}{4} \phi^2 dx},$$

that proves (4). Squaring both sides of (4) gives (5). \square

Proof of Theorem 1. Assume (1) has a positive supersolution u . Then from Proposition 1, c, b must satisfy inequality (5). Let $\gamma > 1$, $R > 2R_0$ and take a smooth function ψ in Ω with $\psi = 0$ for $R_0 < |x| < \frac{R}{2}$ or $|x| > 2\gamma R$, $\psi = 1$ in $R < |x| < \gamma R$, $0 \leq \psi \leq 1$ and $\nabla \psi \leq \frac{4}{R}$. Now we consider $\phi := |x|^{-\beta} \psi$ as a test function in (5), where $\beta := \frac{N-2}{2}$. We have

$$\nabla \phi = -\beta |x|^{-\beta-2} \psi x + |x|^{-\beta} \nabla \psi$$

gives

$$|\nabla \phi|^2 = \beta^2 |x|^{-2\beta-2} \psi^2 - 2\beta |x|^{-2\beta-2} \psi x \cdot \nabla \psi + |x|^{-2\beta} |\nabla \psi|^2,$$

and then by the assumptions on ψ and $\nabla \psi$ we have the estimates

$$|\nabla \phi|^2 \leq \beta^2 |x|^{-N} + \frac{8\beta}{R} |x|^{1-N} + \frac{16}{R^2} |x|^{2-N}, \quad \frac{R}{2} < |x| < R, \text{ or } \gamma R < |x| < 2\gamma R$$

and

$$|\nabla \phi|^2 = \beta^2 |x|^{-N}, \quad R < |x| < \gamma R.$$

Now we write

$$\int_{\Omega} |\nabla \phi|^2 = \int_{\frac{R}{2} < |x| < 2\gamma R} |\nabla \phi|^2 = \int_{\frac{R}{2} < |x| < R} |\nabla \phi|^2 + \int_{R < |x| < \gamma R} |\nabla \phi|^2 + \int_{\gamma R < |x| < 2\gamma R} |\nabla \phi|^2$$

$$:= I_1(R) + I_2(R) + I_3(R).$$

Using the fact that if $\alpha + N \neq 0$ we have

$$\int_{R < |x| < T} |x|^\alpha dx = K_N \int_R^T r^{\alpha+N-1} dr = K_N \frac{T^{\alpha+N} - R^{\alpha+N}}{\alpha + N}$$

and if $\alpha + N = 0$ we have

$$\int_{R < |x| < T} |x|^\alpha dx = K_N \int_R^T r^{\alpha+N-1} dr = K_N \ln \frac{T}{R}$$

(we set $K_N = 1$ as it appears the same in both sides of the inequality) then we compute

$$I_1(R) \leq \beta^2 \ln 2 + 4\beta + 6 := C_N, \quad I_2(R) = \beta^2 \ln \gamma$$

and since $I_3(R) = I_1(2\gamma R)$ we also get

$$I_3(R) \leq C_N.$$

Hence, we proved that

$$\int_{\Omega} |\nabla \phi|^2 \leq 2C_N + \beta^2 \ln \gamma. \quad (23)$$

Recall from (5) we have

$$\int_{\Omega} (c - \frac{b^2}{4}) \phi^2 \leq \int_{\Omega} |\nabla \phi|^2 + 2 \sqrt{\int_{\Omega} |\nabla \phi|^2 \int_{\Omega} \frac{b^2}{4} \phi^2}.$$

We write

$$\int_{\Omega} \frac{b^2}{4} \phi^2 = \int_{\frac{R}{2} < |x| < 2\gamma R} \frac{b^2}{4} \phi^2 \leq \frac{1}{4} \sup_{\frac{R}{2} < |x| < 2\gamma R} |xb(x)|^2 \int_{\frac{R}{2} < |x| < 2\gamma R} \frac{\phi^2}{|x|^2},$$

and estimate the last integral by the Hardy inequality in exterior domains (see [16]) as

$$\int_{\frac{R}{2} < |x| < 2\gamma R} \frac{\phi^2}{|x|^2} \leq \left(\frac{2}{N-2} \right)^2 \int_{\Omega} |\nabla \phi|^2,$$

hence,

$$\int_{\Omega} (c - \frac{b^2}{4}) \phi^2 \leq \left(1 + \frac{2 \sup_{\frac{R}{2} < |x| < 2\gamma R} |xb(x)|}{N-2} \right) \int_{\Omega} |\nabla \phi|^2.$$

The above inequality together inequality (23) give

$$\int_{\Omega} (c - \frac{b^2}{4}) \phi^2 \leq \left(1 + \frac{2 \sup_{\frac{R}{2} < |x| < 2\gamma R} |xb(x)|}{N-2} \right) (\beta^2 \ln 2 + 4\beta + 6 + \beta^2 \ln \gamma),$$

and then using the estimate

$$\int_{\Omega} (c - \frac{b^2}{4}) \phi^2 \geq \int_{R < |x| < \gamma R} (c - \frac{b^2}{4}) |x|^{2\beta} = \int_{R < |x| < \gamma R} |x|^{2\beta-2} |x|^2 (c - \frac{b^2}{4}) \geq \inf_{R < |x| < \gamma R} |x|^2 (c - \frac{b^2}{4}) \ln \gamma,$$

we arrive at

$$\frac{\inf_{R < |x| < \gamma R} |x|^2 (c - \frac{b^2}{4})}{1 + \frac{2 \sup_{\frac{R}{2} < |x| < 2\gamma R} |xb(x)|}{N-2}} \leq \frac{\beta^2 \ln 2 + 4\beta + 6}{\ln \gamma} + \beta^2. \quad (24)$$

Hence, there exists no positive supersolution if inequality (24) violates for some $R > 2R_0$ and $\gamma > 1$, that proves the first part. Also, letting $\gamma \rightarrow \infty$ and then $R \rightarrow \infty$ in (24) we see that (1) does not have any positive supersolution if (7) holds. \square

Proof of Proposition 2. To prove (i) note that from Theorem 1 we have the desired result if

$$\frac{\liminf_{|x| \rightarrow \infty} |x|^2 (c - \frac{b^2}{4})}{\limsup_{|x| \rightarrow \infty} (1 + \frac{2|xb(x)|}{N-2})} > \frac{(N-2)^2}{4}.$$

Now using the fact that

$$\liminf_{|x| \rightarrow \infty} |x|^2 (c - \frac{b^2}{4}) \geq \liminf_{|x| \rightarrow \infty} |x|^2 c - \frac{\tau^2}{4},$$

we see that the inequality above holds if

$$\liminf_{|x| \rightarrow \infty} |x|^2 c > \frac{\tau^2}{4} + \frac{(N-2)^2}{4} (1 + \frac{2\tau}{N-2}) = \frac{(N-2+\tau)^2}{4}.$$

To prove the second part of (i), set $\alpha := \limsup_{|x| \rightarrow \infty} |x|^2 c$, then (9) reads as $\alpha < \frac{(N-2+\tau)^2}{4}$. Now choose $\alpha_1 > \alpha$ and $\tau_1 < \tau$ such that

$$\alpha_1 \leq \frac{(N-2+\tau_1)^2}{4}. \quad (25)$$

Now we look for some $m > 0$ so that the function $u(x) = |x|^{-m}$ is a supersolution of equation (1) in $\mathbb{R}^N \setminus B_{R_1}$ for R_1 sufficiently large. We need

$$-\Delta u + b(x)|\nabla u| - cu = |x|^{-m-2} \left[m(N-2-m) + mb|x| - c|x|^2 \right] \geq 0,$$

for $|x|$ sufficiently large. We have

$$m(N-2-m) + mb|x| - c|x|^2 \geq m(N-2-m) + m\tau_1 - \alpha_1 = -m^2 + (N-2+\tau_1)m - \alpha_1.$$

for $|x|$ sufficiently large, and note the last term is nonnegative for some $m > 0$ if and only if (25) holds.

To prove part (ii), note that if inequality (10) holds then inequality (6) holds with $\gamma = 2$

and the results follows by Theorem 1. \square

Proof of Corollary 1. First note that from Lemma 1, if we take $T = 1$ in (21) we then get

$$\int_{\Omega} |\nabla \phi|^2 dx \geq \int_{\Omega} \frac{-\Delta E}{E} \phi^2 \quad \phi \in C_c^{\infty}(\Omega). \quad (26)$$

Taking $c(x) := \frac{-\Delta E}{E}$ then from (26) we see that this c satisfies (4) with $b \equiv 0$, hence from (1) we must have

$$\liminf_{|x| \rightarrow \infty} |x|^2 c(x) \leq \frac{(N-2)^2}{4},$$

that proves (14). \square

Proof of Corollary 2. By the assumption (18) we have

$$\int_{\Omega} \frac{b^2}{4} \phi^2 \leq \gamma^2 \int_{\Omega} \frac{|\nabla E|^2}{4E^2} \phi^2,$$

and then by the Hardy-type inequality (see [16])

$$\int_{\Omega} |\nabla \phi|^2 \geq \int \frac{|\nabla E|^2}{4E^2} \phi^2 \quad \forall \phi \in C_c^{\infty}(\Omega),$$

we get

$$\int_{\Omega} \frac{b^2}{4} \phi^2 \leq \gamma^2 \int |\nabla \phi|^2.$$

Using this in (5) we get (19). One can now proceed as in the the proof of Theorem 1 to get (20). \square

References

- [1] S. Agmon, On positivity and decay of solutions of second order elliptic equations on Riemannian manifolds, in: Methods of Functional Analysis and Theory of Elliptic Equations, Liguori, Naples, 19-52 (1983).
- [2] S.N. Armstrong and B. Sirakov, Nonexistence of positive supersolutions of elliptic equations via the maximum principle, Comm. Part. Diff. Eqns., 36 (2011), 2011-2047.
- [3] S. Alarcon, J. Garcia-Melian, A. Quaas, Liouville type theorems for elliptic equations with gradient terms, Milan J. Math. 13 81 (2013) 171-185.
- [4] S. Alarcon, J. Garcia-Melian, A. Quaas, Nonexistence of positive supersolutions to some nonlinear elliptic problems, J. Math. Pures Appl. 99 (2013) 618-634.

- [5] S. Alarcon, J. Garcia-Melian, A. Quaas, Existence and non-existence of solutions to elliptic equations with a general convection term, *Proc. Roy. Soc. Edinburgh* 144A (2014) 225-239.
- [6] S. Alarcon, J. Garcia-Melian, A. Quaas, Keller-Osserman type conditions for some elliptic problems with gradient terms, *J. Differential Equations* 252 (2012), 886-914.
- [7] D. Arcoya, C. De Coster, L. Jeanjean, K. Tanaka, Continuum of solutions for an elliptic problem with critical growth in the gradient. *J. Funct. Anal.* 268 (8) (2015), 2298-2335.
- [8] H. Berestycki, F. Hamel, N. Nadirashvili, The speed of propagation for KPP type problems I. Periodic framework, *J. Europ. Math. Soc.* 7 (2005), 173213
- [9] H. Berestycki, F. Hamel, L. Rossi, Liouville type results for semilinear elliptic equations in unbounded domains. *Ann. Mat. Pura Appl.* (4) 186 (2007), 469-507.
- [10] M.A. Burgos-Perez, J. Garcia Melian, A. Quaas, Classification of supersolutions and Liouville theorems for some nonlinear elliptic problems, *Discrete Contin. Dyn. Syst.* (2016). <http://dx.doi.org/10.3934/dcds.2016004>.
- [11] I. Capuzzo Dolcetta, A. Cutri, Hadamard and Liouville type results for fully nonlinear partial differential inequalities, *Commun. Contemp. Math.* 5 (2003), no. 3, 435448.
- [12] G. Caristi, E. Mitidieri, Nonexistence of positive solutions of quasilinear equations. *Adv. Diff. Equ.* 2 (1997), 317-359.
- [13] H. Chen, P. Felmer, On Liouville type theorems for fully nonlinear elliptic equations with gradient term, *J. Differential Equations* 255 (2013) 2167-2195.
- [14] H. Chen, R. Peng, F. Zhou, Nonexistence of Positive Supersolution to a Class of Semilinear Elliptic Equations and Systems in an Exterior Domain, available at <https://arxiv.org/abs/1803.02531>
- [15] H. Chen, A. Quaas and F. Zhou, On nonhomogeneous elliptic equations with the Hardy-Leray potentials, *arXiv:1705.08047*.
- [16] C. Cowan, Optimal Hardy inequalities for general elliptic operators with improvements, *Commun. Pure Appl. Anal.* 9 (2010), no. 1, 109-140.
- [17] B. Devyver, M. Fraas and Y. Pinchover, Optimal Hardy Weight for Second-Order Elliptic Operator: an answer to a problem of Agmon, *J. Functional Analysis* 266, 4422-4489 (2014).
- [18] P. Felmer, A. Quaas, B. Sirakov, Solvability of nonlinear elliptic equations with gradient terms. *J. Diff. Eq.* 254 (11) (2013), 4327-4346.
- [19] L. Jeanjean, B. Sirakov, Existence and multiplicity for elliptic problems with quadratic growth in the gradient. *Comm. Part. Diff. Eq.* 38 (2013), 244-264.

- [20] Y. Pinchover, A Liouville-type theorem for Schrödinger operators, *Commun. Math. Phys.* 272(1), 75-84 (2007).
- [21] Y. Pinchover, Topics in the theory of positive solutions of second-order elliptic and parabolic partial differential equations, In *Proceedings of Symposia in Pure Mathematics* (Vol. 76, No. 1, p. 329). Providence, RI; American Mathematical Society; 1998.
- [22] L. Rossi, Non-existence of positive solutions of fully nonlinear elliptic bounded domains, *Commun. Pure Appl. Anal.* 7 (2008), 125-141.
- [23] J. Serrin and H. Zou, Cauchy-Liouville and universal boundedness theorems for quasilinear elliptic equations and inequalities, *Acta. Math.* 189 (2002), 79-142.