# Regularity of stable solutions of a Lane-Emden type system

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**Abstract**

We examine the system given by

$$\left\{ \begin{array}{rcll} -\Delta u &=& \lambda (v+1)^p & \Omega \\ -\Delta v &=& \gamma (u+1)^\theta & \Omega, \\ u &=& v=0 & \partial\Omega, \end{array} \right.$$

where $\lambda, \gamma$ are positive parameters and where $1 < p \le \theta$ and where $\Omega$ is a smooth bounded domain in $\mathbb{R}^N$. We show the extremal solutions associated with the above system are bounded provided

$$\frac{N}{2} < 1 + \frac{2(\theta+1)}{p\theta - 1} \left( \sqrt{\frac{p\theta(p+1)}{\theta+1}} + \sqrt{\frac{p\theta(p+1)}{\theta+1} - \sqrt{\frac{p\theta(p+1)}{\theta+1}}} \right).$$

In particular this shows that the extremal solutions are bounded for any $1 < p \le \theta$ provided $N \le 10$.

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# 1 Introduction

In this work we examine the following system:

$$(N)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u &= \lambda(v+1)^p \qquad \Omega \\ -\Delta v &= \gamma(u+1)^\theta \qquad \Omega, \\ u &= v = 0 \qquad \partial\Omega, \end{cases}$$

where $\Omega$ is a bounded domain in $\mathbb{R}^N$, $\lambda, \gamma > 0$ are positive parameters and where $p, \theta > 1$. Our interest is in the regularity of the extremal solutions associated with $(N)_{\lambda,\gamma}$. In particular we are interested when the extremal solutions of $(N)_{\lambda,\gamma}$ are bounded, since one can then apply elliptic regularity theory to show the extremal solutions are classical solutions. The main approach will be to utilize the stability of the extremal solutions to obtain added regularity. We remark that for this system, when $p \neq \theta$, it is not clear how to utilize the stability in a meaningful way and this work represents a first work in this direction. The nonlinearities we examine naturally fit into the following class:

(R): $f$ is smooth, increasing, convex on $\mathbb{R}$ with $f(0) = 1$ and $f$ is superlinear at $\infty$ (i.e. $\lim_{u\to\infty} \frac{f(u)}{u} = \infty$).

## 1.1 Second order scalar case

For a nonlinearity $f$ of type (R) consider the following second order scalar analog of the above system given by

$$(Q)_\lambda \qquad \begin{cases} -\Delta u = \lambda f(u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

This scalar equation is now quite well understood whenever $\Omega$ is a bounded smooth domain in $\mathbb{R}^N$. See, for instance, [1, 2, 3, 4, 14, 15, 17, 18, 19, 21]. We now list the properties one comes to expect when studying $(Q)_\lambda$.

- There exists a finite positive critical parameter $\lambda^*$, called the **extremal parameter**, such that for all $0 < \lambda < \lambda^*$ there exists a a smooth **minimal solution** $u_\lambda$ of $(Q)_\lambda$. By minimal solution, we mean here that if $v$ is another solution of $(Q)_\lambda$ then $v \geq u_\lambda$ a.e. in $\Omega$.

- For each $0 < \lambda < \lambda^*$ the minimal solution $u_\lambda$ is **semi-stable** in the sense that

$$\int_\Omega \lambda f'(u_\lambda)\psi^2 dx \leq \int_\Omega |\nabla\psi|^2 dx, \qquad \forall \psi \in H_0^1(\Omega),$$

and is unique among all the weak semi-stable solutions.

- The map $\lambda \mapsto u_\lambda(x)$ is increasing on $(0, \lambda^*)$ for each $x \in \Omega$. This allows one to define $u^*(x) := \lim_{\lambda \nearrow \lambda^*} u_\lambda(x)$, the so-called **extremal solution**, which can be shown to be a weak solution of $(Q)_{\lambda^*}$. In addition one can show that $u^*$ is the unique weak solution of $(Q)_{\lambda^*}$. See [18].

- There are no solutions of $(Q)_\lambda$ (even in a very weak sense) for $\lambda > \lambda^*$.

A question which has attracted a lot of attention is whether the extremal function $u^*$ is a classical solution of $(Q)_{\lambda^*}$. This is of interest since one can then apply the results from [11] to start a second branch of solutions emanating from $(\lambda^*, u^*)$. The answer typically depends on the nonlinearity $f$, the dimension $N$ and the geometry of the domain $\Omega$. We now list some known results.

- [11] Suppose $f(u) = e^u$. If $N < 10$ then $u^*$ is bounded. For $N \geq 10$ and $\Omega$ the unit ball $u^*(x) = -2\log(|x|)$.

- [4] Suppose $f$ satisfies (R) but without the convexity assumption and $\Omega$ is the unit ball. Then $u^*$ is bounded for $N < 10$. In view of the above result this is optimal.

- On general domains, and if $f$ satisfies (R), then $u^*$ is bounded for $N \leq 3$ [21]. Recently this has been improved to $N \leq 4$ provided the domain is convex (again one can drop the convexity assumption on $f$), see [3].

We now examine the generalization of $(N)_{\lambda,\gamma}$ given by

$$(P)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u &= \lambda f(v) & \Omega \\ -\Delta v &= \gamma g(u) & \Omega, \\ \quad u &= v = 0 & \partial\Omega, \end{cases}$$

where $f$ and $g$ satisfy (R). Define $\mathcal{Q} = \{(\lambda, \gamma) : \lambda, \gamma > 0\}$,

$$\mathcal{U} := \{(\lambda, \gamma) \in \mathcal{Q} : \text{ there exists a smooth solution } (u, v) \text{ of } (P)_{\lambda,\gamma}\},$$

and set $\Upsilon := \partial\mathcal{U} \cap \mathcal{Q}$. Note that $\Upsilon$ is the analog of $\lambda^*$ for the above system. A generalization of $(P)_{\lambda,\gamma}$ was examined in [20] and many results were obtained, including

**Theorem.** *(Montenegro, [20]) Suppose $f$ and $g$ satisfy (R). Then*

1. *$\mathcal{U}$ is nonempty.*
2. *For all $(\lambda,\gamma)\in\mathcal{U}$ there exists a smooth, minimal solution of $(P)_{\lambda,\gamma}$.*
3. *For each $0<\sigma<\infty$ there is some $0<\lambda_\sigma^*<\infty$ such that $\mathcal{U}\cap\{(\lambda,\sigma\lambda): 0<\lambda\}$ is given by $\{(\lambda,\sigma\lambda): 0<\lambda<\lambda_\sigma^*\}\cup\mathcal{H}$ where $\mathcal{H}$ is either the empty set or $\{(\lambda_\sigma^*,\sigma\lambda_\sigma^*)\}$. The map $\sigma\mapsto\lambda_\sigma^*$ is bounded on compact subsets of $(0,\infty)$. Fix $0<\sigma<\infty$ and let $(u_\lambda,v_\lambda)$ denote the smooth minimal solution of $(P)_{\lambda,\sigma\lambda}$ for $0<\lambda<\lambda_\sigma^*$. Then $u_\lambda(x),v_\lambda(x)$ are increasing in $\lambda$ and hence*

$$u^*(x):=\lim_{\lambda\nearrow\lambda_\sigma^*}u_\lambda(x),\qquad v^*(x):=\lim_{\lambda\nearrow\lambda_\sigma^*}u_\lambda(x),$$

*are well defined and can be shown to be a weak solution of $(P)_{\lambda_\sigma^*,\sigma\lambda_\sigma^*}$.*

Our notation will vary slightly from above. Let $(\lambda^*,\gamma^*)\in\Upsilon$ and set $\sigma:=\frac{\gamma^*}{\lambda^*}$. Define $\Gamma_\sigma:=\{(\lambda,\sigma\lambda):\frac{\lambda^*}{2}<\lambda<\lambda^*\}$ and we let $(u^*,v^*)$, called the **extremal solution** associated with $(P)_{\lambda^*,\gamma^*}$, be the pointwise limit of the minimal solutions along the ray $\Gamma_\sigma$ as $\lambda\nearrow\lambda^*$. As mentioned above $(u^*,v^*)$ is a weak solution of $(N)_{\lambda^*,\gamma^*}$ in a suitable sense.

The following result shows that the minimal solutions are stable in some suitable sense and this will be crucial in obtaining regularity of the extremal solutions associated with $(N)_{\lambda,\gamma}$.

**Theorem.** *(Montenegro [20]) Let $(\lambda,\gamma)\in\mathcal{U}$ and let $(u,v)$ denote the minimal solution of $(P)_{\lambda,\gamma}$. Then $(u,v)$ is semi-stable in the sense that there is some smooth $0<\zeta,\chi\in H_0^1(\Omega)$ and $0\le\eta$ such that*

$$-\Delta\zeta=\lambda f'(v)\chi+\eta\zeta,\qquad -\Delta\chi=\gamma g'(u)\zeta+\eta\chi,\qquad \textit{in } \Omega. \tag{1}$$

We give an alternate proof of a result which is slightly different than the above one, but which is sufficient for our purposes. Fix $(\lambda^*,\gamma^*)\in\Upsilon$, $\sigma:=\frac{\gamma^*}{\lambda^*}$ and let $(u_\lambda,v_\lambda)$ denote minimal solution of $(P)_{\lambda,\gamma}$ on the ray $\Gamma_\sigma$. Taking a derivative in $\lambda$ of $(P)_{\lambda,\sigma\lambda}$ shows that

$$-\Delta\tilde{\zeta}=\lambda f'(v_\lambda)\tilde{\chi}+f(v_\lambda),\quad -\Delta\tilde{\chi}=\lambda\sigma g'(u_\lambda)\tilde{\zeta}+\sigma g(u_\lambda)\quad \text{ in } \Omega,$$

where $\tilde{\zeta}:=\partial_\lambda u_\lambda$ and $\tilde{\chi}:=\partial_\lambda v_\lambda$. Using the monotonicity of $u_\lambda,v_\lambda$ and the maximum principle shows that $\tilde{\zeta},\tilde{\chi}>0$.

We now recall some known results regarding the regularity of the extremal solutions associated with various systems. In what follows $\Omega$ is a bounded domain in $\mathbb{R}^N$.

- In [7] the following system

$$(E)_{\lambda,\gamma} \qquad \begin{cases} -\Delta u &= \lambda e^v & \Omega \\ -\Delta v &= \gamma e^u & \Omega, \\ \quad u &= v = 0 & \partial\Omega, \end{cases}$$

was examined. It was shown that if $3 \leq N \leq 9$ and

$$\frac{N-2}{8} < \frac{\gamma^*}{\lambda^*} < \frac{8}{N-2},$$

then the extremal solution $(u^*, v^*)$ is bounded. Note that not only does the dimension $N$ play a role but how close $(\lambda^*, \gamma^*)$ are to the diagonal $\gamma = \lambda$ plays a role. When $\gamma = \lambda$ one can show that the above system reduces to the scalar equation $-\Delta u = \lambda e^u$. We remark that we were unable to extend the methods used in [7] to handle $(N)_{\lambda,\gamma}$ except in the case where $p = \theta$.

- In [9] the system

$$(P')_{\lambda,\gamma} \qquad -\Delta u = \lambda F(u,v), \qquad -\Delta v = \gamma G(u,v) \qquad \text{in } \Omega,$$

with $u = v = 0$ on $\partial\Omega$ was examined examined in the cases where $F(u,v) = f'(u)g(v),\ G(u,v) = f(u)g'(v)$ (resp. $F(u,v) = f(u)g'(v),\ G(u,v) = f'(u)g(v)$) and were denoted by $(G)_{\lambda,\gamma}$ (resp. $(H)_{\lambda,\gamma}$). It was shown that the extremal solutions associated with $(G)_{\lambda,\gamma}$ were bounded provided $\Omega$ was a convex domain in $R^N$ where $N \leq 3$ and $f$ and $g$ satisfied conditions similar to (R). Regularity results regarding $(H)_{\lambda,\gamma}$ we also obtained in the case where at least one of of $f$ and $g$ were explicit nonlinearities given by $(u+1)^p$ or $e^u$.

**Remark 1.** 1. *After the completion of this work we were made aware of a recent improvement of the results from [7]. In [13] they examined $(E)_{\lambda,\gamma}$ and showed the extremal solutions are bounded provided $N \leq 9$. They make no assumptions on the parameters $\lambda, \gamma$. In their approach they obtain, independent of our work, the stability inequality (4). We also mention a related work [10] which examined fourth order scalar problems but used similar techniques.*

2. *Since there was considerable delay from the completion of this paper to publication, we now mention various works that have appeared in the meantime. The ideas of this work were extended in [6] to prove various Liouville theorems regarding solutions of $-\Delta u = v^p$ and $-\Delta v = u^\theta$ in*

*$\mathbb{R}^N$ and $\mathbb{R}^N_+$. The methods developed here and then extended to Liouville theorems in [6] were then extended to handle some Liouville theorems related to some Hénon systems in $\mathbb{R}^N$ in [16] (this work also contains many other results). Regarding Lane-Emden system we finally mention the work [5] which obtained optimal Liouville theorems related to the Lane-Emden system $-\Delta u = v^p$, $-\Delta v = u^\theta$ in $\mathbb{R}^N$, at least in the case of radial solutions. We finally mention a very powerful new monotonicity method which was developed in [12] to prove Liouville theorems related to stable solutions of $\Delta^2 u = |u|^{p-1}u$ in $\mathbb{R}^N$. Since that work [12] work there has been many extensions of that method.*

## 2 Main Results

We now state our main results.

**Theorem 1.** *Suppose that $1 < p \le \theta$, $(\lambda^*, \gamma^*) \in \Upsilon$ and let $(u^*, v^*)$ denote the extremal solution associated with $(N)_{\lambda^*,\gamma^*}$. Suppose that*

$$\frac{N}{2} < 1 + \frac{2(\theta+1)}{p\theta - 1}\left(\sqrt{\frac{p\theta(p+1)}{\theta+1}} + \sqrt{\frac{p\theta(p+1)}{\theta+1} - \sqrt{\frac{p\theta(p+1)}{\theta+1}}}\right). \tag{2}$$

*Then $u^*, v^*$ are bounded.*

**Remark 2.** *We are interested in obtaining lower bounds on the right hand side of (2), in the case where $1 < p \le \theta$, and so we set $f(p,\theta) := \frac{4(\theta+1)}{p\theta-1} t_0$, where*

$$t_0 := \left(\sqrt{\frac{p\theta(p+1)}{\theta+1}} + \sqrt{\frac{p\theta(p+1)}{\theta+1} - \sqrt{\frac{p\theta(p+1)}{\theta+1}}}\right).$$

*Note that (2) is satisfied exactly when $N - 2 < f(p,\theta)$. We now rewrite $f$ using the change of variables $z = \frac{p\theta(p+1)}{\theta+1}$ to arrive at*

$$\tilde{f}(p,z) = \frac{4p}{z-p}\left(\sqrt{z} + \sqrt{z - \sqrt{z}}\right),$$

*and the transformed domain is given by*

$$\mathcal{D} = \{(p,z) : p \ge 1,\ p^2 \le z \le p^2 + p\}.$$

*A computer algebra system easily shows that $\tilde{f} > 8$ on $\partial\mathcal{D}$. Also note that $\partial_p \tilde{f} > 0$ on $\mathcal{D}$ and so we see that $\tilde{f} > 8$ on $\mathcal{D}$ and hence we see that the extremal solutions are bounded for any $1 < p \le \theta$ provided $N \le 10$.*

There are two main steps in proving the above theorems. We first show that minimal solutions of $(P)_{\lambda,\gamma}$, which are semi-stable in the sense of (1), satisfy a stability inequality which is reminiscent of semi-stability in the sense of the second order scalar equations. This is given by Lemma 1.

The second ingredient will be a pointwise comparison result between $u$ and $v$, given in Lemma 2. We remark that this was motivated by [22] and a similar result was used in [8].

**Lemma 1.** *Let $(u,v)$ denote a semi-stable solution of $(P)_{\lambda,\gamma}$ in the sense of (1). Then*

$$2\sqrt{\lambda\gamma}\int \sqrt{f'(v)g'(u)}\phi\psi \leq \int |\nabla\phi|^2 + |\nabla\psi|^2, \tag{3}$$

*for all $\phi,\psi \in H_0^1(\Omega)$. Taking $\phi=\psi$ gives*

$$\sqrt{\lambda\gamma}\int \sqrt{f'(v)g'(u)}\phi^2 \leq \int |\nabla\phi|^2 \tag{4}$$

*for all $\phi \in H_0^1(\Omega)$.*

*Proof.* Since $(u,v)$ is a semi-stable solution of $(P)_{\lambda,\gamma}$ there is some $0<\zeta,\chi \in H_0^1(\Omega)$ smooth such that

$$\frac{-\Delta\zeta}{\zeta} \geq \lambda f'(v)\frac{\chi}{\zeta}, \qquad \frac{-\Delta\chi}{\chi} \geq \gamma g'(u)\frac{\zeta}{\chi}, \qquad \text{in } \Omega.$$

Let $\phi,\psi \in C_c^\infty(\Omega)$ and multiply the first equation by $\phi^2$ and the second by $\psi^2$ and integrate over $\Omega$ to arrive at

$$\int \lambda f'(v)\frac{\chi}{\zeta}\phi^2 \leq \int |\nabla\phi|^2, \qquad \int \gamma g'(u)\frac{\zeta}{\chi}\psi^2 \leq \int |\nabla\psi|^2,$$

where we have utilized the result that for any sufficiently smooth $E>0$ we have

$$\int \frac{-\Delta E}{E}\phi^2 \leq \int |\nabla\phi|^2,$$

for all $\phi \in C_c^\infty(\Omega)$. We now add the inequalities to obtain

$$\int (\lambda f'(v)\phi^2)\frac{\chi}{\zeta} + (\gamma g'(u)\psi^2)\frac{\zeta}{\chi} \leq \int |\nabla\phi|^2 + |\nabla\psi|^2. \tag{5}$$

Now note that

$$2\sqrt{\lambda\gamma f'(v)g'(u)}\phi\psi \leq 2t\lambda f'(v)\phi^2 + \frac{1}{2t}\gamma g'(u)\psi^2,$$

for any $t > 0$. Taking $2t = \frac{\chi(x)}{\zeta(x)}$ gives

$$2\sqrt{\lambda\gamma f'(v)g(u)}\phi\psi \leq (\lambda f'(v)\phi^2)\frac{\chi}{\zeta} + (\gamma g'(u)\psi^2)\frac{\zeta}{\chi},$$

and putting this back into (5) gives the desired result.

□

**Lemma 2.** *Let $(u,v)$ denote a smooth solution of $(N)_{\lambda,\gamma}$ and suppose that $\theta \geq p > 1$. Define*

$$\alpha := \max\left\{0, \left(\frac{\gamma(p+1)}{\lambda(\theta+1)}\right)^{\frac{1}{p+1}} - 1\right\}.$$

*Then*

$$\lambda(\theta+1)(v+1+\alpha)^{p+1} \geq \gamma(p+1)(u+1)^{\theta+1} \quad \text{in } \Omega. \tag{6}$$

*Proof.* Let $(u,v)$ denote a smooth solution of $(N)_{\lambda,\gamma}$ and define $w := v + 1 + \alpha - C(u+1)^t$ where

$$C := \left(\frac{\gamma(p+1)}{\lambda(\theta+1)}\right)^{\frac{1}{p+1}} \quad \text{and} \quad t := \frac{\theta+1}{p+1} \geq 1.$$

Note that $w \geq 0$ on $\partial\Omega$ and define $\Omega_0 := \{x \in \Omega : w(x) < 0\}$. If $\Omega_0$ is empty then we are done so we suppose that $\Omega_0$ is nonempty. Note that since $w \geq 0$ on $\partial\Omega$ we have $w = 0$ on $\partial\Omega_0$. A computation shows that

$$-\Delta w = \gamma(u+1)^\theta - Ct(u+1)^{t-1}\lambda(v+1)^p + Ct(t-1)(u+1)^{t-2}|\nabla u|^2, \qquad \text{in } \Omega,$$

and since $t \geq 1$ we have

$$-\Delta w \geq \gamma(u+1)^\theta - Ct(u+1)^{t-1}\lambda(v+1)^p \quad \text{in } \Omega.$$

Note that we have, by definition,

$$v+1 \leq v+1+\alpha < C(u+1)^t \qquad \text{in } \Omega_0,$$

and so we have

$$-\Delta w \geq \gamma(u+1)^\theta - C^{p+1}t\lambda(u+1)^{tp+t-1} \quad \text{in } \Omega_0,$$

but the right hand side of this is zero and hence we have $-\Delta w \geq 0$ in $\Omega_0$ with $w = 0$ on $\partial\Omega_0$ and hence $w \geq 0$ in $\Omega_0$, which is a contradiction. So $\Omega_0$ is empty.

□

**Proof of Theorem 1.** Let $(\lambda^*, \gamma^*) \in \Upsilon$ and let $\sigma := \frac{\gamma^*}{\lambda^*}$ and suppose that $(u,v)$ denotes a minimal solution of $(N)_{\lambda,\gamma}$ on the ray $\Gamma_\sigma$. Put $\phi := (v+1)^t - 1$, where $\frac{1}{2} < t$, into (4) to obtain

$$\sqrt{\lambda\gamma p\theta} \int (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta-1}{2}} ((v+1)^t - 1)^2 \leq t^2 \int (v+1)^{2t-2} |\nabla v|^2,$$

and multiply $(N)_{\lambda,\gamma}$ by $(v+1)^{2t-1} - 1$ and integrate by parts to obtain

$$t^2 \int (v+1)^{2t-2} |\nabla v|^2 = \frac{t^2\gamma}{2t-1} \int (u+1)^\theta ((v+1)^{2t-1} - 1).$$

Equating these and expanding the squares and dropping some positive terms gives

$$\begin{aligned} &\sqrt{\lambda\gamma p\theta} \int (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta-1}{2}} (v+1)^{2t} \\ &\quad \leq \frac{t^2\gamma}{2t-1} \int (u+1)^\theta (v+1)^{2t-1} \\ &\qquad + 2\sqrt{\lambda\gamma p\theta} \int (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta-1}{2}} (v+1)^t. \end{aligned} \tag{7}$$

We now use Lemma 2 to get a lower bound for

$$I := \int (v+1)^{\frac{p-1}{2}} (u+1)^{\frac{\theta-1}{2}} (v+1)^{2t},$$

but we need to rework the pointwise estimate (6) first. From (6) we have

$$\sqrt{\frac{\gamma(p+1)}{\lambda(\theta+1)}} (u+1)^{\frac{\theta+1}{2}} \leq (v+1+\alpha)^{\frac{p+1}{2}},$$

and for all $\delta > 0$ there is some $C(\delta) > 0$ such that

$$(v+1+\alpha)^{\frac{p+1}{2}} \leq (1+\delta)(v+1)^{\frac{p+1}{2}} + C(\delta)\alpha^{\frac{p+1}{2}}.$$

From this we see that there is some $C_1 = C_1(\delta, p, \alpha)$ such that

$$(v+1)^{\frac{p+1}{2}} \geq \sqrt{\frac{\gamma(p+1)}{\lambda(\theta+1)}} \frac{(u+1)^{\frac{\theta+1}{2}}}{1+\delta} - C_1.$$

We now rewrite $I$ as

$$I = \int (u+1)^{\frac{\theta-1}{2}} (v+1)^{2t-1} (v+1)^{\frac{p+1}{2}},$$

and use the above estimate to show that

$$I \geq \sqrt{\frac{\gamma(p+1)}{\lambda(\theta+1)}} \frac{1}{\delta+1} \int (u+1)^{\theta}(v+1)^{2t-1} - C_1 \int (u+1)^{\frac{\theta-1}{2}}(v+1)^{2t-1}. \quad (8)$$

We now return to (7) and write the left hand side, where $\varepsilon > 0$ is small, as

$$\varepsilon\sqrt{\lambda\gamma p\theta} I + (1-\varepsilon)\sqrt{\lambda\gamma p\theta} I,$$

and we leave the first term alone and we use the above lower estimate for $I$ on the second term. Putting this back into (7) and after some rearranging one arrives at

$$\begin{aligned} \varepsilon\sqrt{\lambda\gamma p\theta} I \quad &+ \quad \gamma K \int (u+1)^{\theta}(v+1)^{2t-1} \\ &\leq \quad 2\sqrt{\lambda\gamma p\theta}\, I_1 + (1-\varepsilon)\sqrt{\lambda\gamma p\theta} C_1\, I_2 \end{aligned} \quad (9)$$

where

$$K := \frac{(1-\varepsilon)}{1+\delta}\sqrt{\frac{p\theta(p+1)}{\theta+1}} - \frac{t^2}{2t-1},$$

$$I_1 := \int (v+1)^{\frac{p-1}{2}}(u+1)^{\frac{\theta-1}{2}}(v+1)^{t}, \qquad \text{and}$$

$$I_2 := \int (u+1)^{\frac{\theta-1}{2}}(v+1)^{2t-1}.$$

For the moment we assume the following **claims:** for all $T > 1$ and $k > 1$

$$\begin{aligned} I_2 \quad &\leq \quad \frac{1}{T^{\frac{\theta+1}{2}}} \int (u+1)^{\theta}(v+1)^{2t-1} + |\Omega| T^{\frac{\theta-1}{2}} k^{2t-1} \\ &\quad + \frac{1}{k^{\frac{p+1}{2}}} \int (u+1)^{\frac{\theta-1}{2}}(v+1)^{\frac{p-1}{2}+2t}, \end{aligned} \quad (10)$$

and

$$\begin{aligned} I_1 \quad &\leq \quad \frac{1}{T^{t}} \int (u+1)^{\frac{\theta-1}{2}}(v+1)^{\frac{p-1}{2}+2t} + |\Omega| T^{\frac{p-1}{2}+t} k^{\frac{\theta-1}{2}} \\ &\quad + \frac{T^{|\frac{p+1}{2}-t|}}{k^{\frac{\theta+1}{2}}} \int (u+1)^{\theta}(v+1)^{2t-1}. \end{aligned} \quad (11)$$

Putting (10) and (11) back into (9) one arrives at an estimate of the form

$$K_1 I + K_2 \int (u+1)^{\theta}(v+1)^{2t-1} \leq C(\varepsilon, p, \theta, T, k, \delta) \quad (12)$$

where

$$K_1 := \varepsilon\sqrt{\lambda\gamma p\theta} - \frac{2\sqrt{\lambda\gamma p\theta}}{T^t} - \frac{(1-\varepsilon)\sqrt{\lambda\gamma p\theta}\, C_1}{k^{\frac{p+1}{2}}}, \qquad \text{and}$$

$$K_2 := \gamma K - \frac{2\sqrt{\lambda\gamma p\theta}\, T^{|\frac{p+1}{2}-t|}}{k^{\frac{\theta+1}{2}}} - \frac{(1-\varepsilon)\sqrt{\lambda\gamma p\theta}\, C_1}{T^{\frac{\theta+1}{2}}},$$

and where $C(\varepsilon, p, \theta, T, k, \delta)$ is a positive finite constant which is uniform on the ray $\Gamma_\sigma$. Define

$$t_0 := \sqrt{\frac{p\theta(p+1)}{\theta+1}} + \sqrt{\frac{p\theta(p+1)}{\theta+1} - \sqrt{\frac{p\theta(p+1)}{\theta+1}}}$$

and note that $t_0 > 1$ for all $p, \theta > 1$. Fix $1 < t < t_0$ and hence

$$\sqrt{\frac{p\theta(p+1)}{\theta+1}} - \frac{t^2}{2t-1} > 0.$$

We now fix $\varepsilon > 0$ and $\delta > 0$ sufficiently small such that $K > 0$. We now fix $T > 1$ sufficiently large such that

$$\varepsilon\sqrt{\lambda\gamma p\theta} - \frac{2\sqrt{\lambda\gamma p\theta}}{T^t}$$

(this is the first two terms from $K_1$) and

$$\gamma K - \frac{(1-\varepsilon)\sqrt{\lambda\gamma p\theta}\, C_1}{T^{\frac{\theta+1}{2}}}$$

(the first and third terms from $K_2$) are positive and bounded away from zero on the the ray $\Gamma_\sigma$. We now take $k > 1$ sufficiently big such that $K_1, K_2$ are positive and bounded away from zero on the ray $\Gamma_\sigma$ and hence we have estimates of the form: for all $1 < t < t_0$ there is some $C_t > 0$ such that

$$\int (u+1)^\theta (v+1)^{2t-1} \leq C_t, \tag{13}$$

where $C_t$ is some finite uniform constant on the ray $\Gamma_\sigma$. Using the pointwise lower estimate (6) for $v+1$ gives: for all $1 < t < t_0$ there is some $\tilde{C}_t < \infty$, uniform along the ray $\Gamma_\sigma$, such that

$$\int (u+1)^{\theta + (\frac{(\theta+1)(2t-1)}{p+1}} \leq \tilde{C}_t, \tag{14}$$

and hence this estimate also holds if one replaces $u$ with $u^*$. We now let $1 < t < t_0$ and note that

$$\frac{1}{\gamma}\int |\nabla v|^2 = \int (u+1)^\theta v \leq \int (u+1)^\theta (v+1) \leq \int (u+1)^\theta (v+1)^{2t-1} \leq C_t,$$

by (13) and hence we can pass to the limit and see that $v^* \in H_0^1(\Omega)$. We now proceed to show that $v^*$ is bounded in low dimensions. First note that

$$\frac{-\Delta v^*}{\gamma^*} = (u^*+1)^\theta = \frac{(u^*+1)^\theta}{v^*+1} v^* + \frac{(u^*+1)^\theta}{v^*+1} \quad \text{in } \Omega.$$

To show that $v^*$ is bounded it is sufficient, since $v^* \in H_0^1(\Omega)$, to show that $\frac{(u^*+1)^\theta}{v^*+1} \in L^T(\Omega)$ for some $T > \frac{N}{2}$. Using (6) and passing to the limit one sees there is some $C > 0$ such that

$$\frac{(u^*+1)^\theta}{v^*+1} \leq C(u^*+1)^{\frac{p\theta-1}{p+1}} + C \qquad \text{in } \Omega,$$

and so $\frac{(u^*+1)^\theta}{v^*+1} \in L^T(\Omega)$ for some $T > \frac{N}{2}$ provided

$$\frac{(p\theta-1)}{p+1}\frac{N}{2} < \theta + \frac{(\theta+1)(2t_0-1)}{p+1},$$

after considering (14). This rearranges into

$$\frac{N}{2} < 1 + 2\frac{(\theta+1)}{p+1} t_0,$$

which is the desired result. We now use $(N)_{\lambda,\gamma}$ and elliptic regularity to see that $u^*$ is also bounded.

□

**Proof of Claims (10) and (11).** We first prove (10). We write $I_2$ as

$$I_2 = \int_{u+1 \geq T} + \int_{u+1<T, v+1 \leq k} + \int_{u+1<T, v+1>k},$$

where the integrands are the same as in $I_2$. Note that the first integral is less than or equal

$$\int_{u+1>T} (u+1)^{\frac{\theta-1}{2}} \left(\frac{u+1}{T}\right)^{\frac{\theta+1}{2}} (v+1)^{2t-1} \leq \frac{1}{T^{\frac{\theta+1}{2}}} \int (u+1)^\theta (v+1)^{2t-1}.$$

The second integral is trivial to get upper estimate on. One estimates the third integral in the same way as the first to see that

$$\int_{u+1<T, v+1>k} \leq \frac{1}{k^{\frac{p+1}{2}}} \int (u+1)^{\frac{\theta-1}{2}} (v+1)^{\frac{p-1}{2}+2t}.$$

Combining these estimates gives (10). We now prove (11). We write

$$I_1 = \int_{v+1\geq T} + \int_{v+1<T, u+1<k} + \int_{v+1<T, u+1\geq k},$$

where the integrands are the same as $I_1$. Note that the first integral is less than or equal

$$\int_{v+1>T} (v+1)^{\frac{p-1}{2}+t} \left(\frac{v+1}{T}\right)^t (u+1)^{\frac{\theta-1}{2}},$$

and this is less than or equal

$$\frac{1}{T^t} \int (v+1)^{\frac{p-1}{2}+2t} (u+1)^{\frac{\theta-1}{2}}.$$

The second integral is easily estimated. We rewrite the third integral as

$$\int_{v+1<T, u+1\geq k} (v+1)^{2t-1} (u+1)^{\theta} \left( (v+1)^{\frac{p+1}{2}-t} (u+1)^{\frac{-\theta-1}{2}} \right),$$

and we now estimate the terms inside the bracket in the obvious manner. Combining these gives (11).

□

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