Math 7460 HW2 Sol.

Fall 2019

(i)

$$f^{\varepsilon}(x) - f(x) = \int_{\mathbb{R}} \phi_\varepsilon(y) f(x-y)\, dy - f(x)$$

$$= \int_{\mathbb{R}} \phi_\varepsilon(y) \left[f(x-y) - f(x) \right] dy$$

$$= \int_{|y|<\varepsilon} \phi_\varepsilon(y) \left[f(x-y) - f(x) \right] dy.$$

let $\delta > 0$, $\exists\, \varepsilon_\delta > 0$ s.t.

$|f(x-y) - f(x)| < \delta$ $\quad \forall x \in \mathbb{R}$, $\forall |y| < \varepsilon_\delta$.

let $0 < \varepsilon < \varepsilon_\delta$, then

$$|f^\varepsilon(x) - f(x)| \le \int_{|y|<\varepsilon} \varphi_\varepsilon(y) \, |f(x-y) - f(x)| \, dy$$

$$\le \delta \int_{|y|<\varepsilon} \varphi_\varepsilon(y) \, dy$$

$$= \delta.$$

So this shows $f^\varepsilon \to f$ unif.

in $\mathbb{R}$.

(ii)

$\|T_y f - T_y g\|_{L^p}$

$= \|T_y(f-g)\|_{L^p}$ (since T_y linear).

$= \|f-g\|_{L^p}$ (shifting doesn't effect norm)

$< \delta$.

Now show $\lim_{y\to 0} \sup \|T_y(g) - g\|_{L^p} = 0$.

~~See~~ $T_y g(x) - g(x) = g(x-y) - g(x)$

and $g \in C_c(\mathbb{R})$ so we see

$\sup_{x\in\mathbb{R}} |T_y g(x) - g(x)| \longrightarrow 0$ as $y \to 0$

Hence same for L^p norm

$\boxed{\text{~~False if g NOT compactly supported~~}}$

where we are using fact g has compact support.

Final Part

$$\| T_y f - f \|_{L^p} \leq 2\delta + \| T_y g - g \|_{L^p}$$

$$\Rightarrow \overline{\lim_{y \to 0}} \| T_y f - f \|_{L^p} \leq 2\delta + 0$$

But $\delta > 0$ arbitrary.

we Have

$$|f^{\varepsilon}(x) - f(x)| \leq \int_{\mathbb{R}} |f(x-y) - f(x)| \, d\mu_\varepsilon(y).$$

$d\mu_\varepsilon(y) = \phi_\varepsilon(y)\, dy$

So $\int_{\mathbb{R}} d\mu_\varepsilon(y) = 1$ so we

Apply Jensen

$$|f^{\varepsilon}(x) - f(x)|^p \leq \int_{\mathbb{R}} |f(x-y) - f(x)|^p \, d\mu_\varepsilon(y).$$

Integrate in x

$$\int_{\mathbb{R}} |f^{\varepsilon}(x) - f(x)|^p \, dx$$

$$\leq \int_{\mathbb{R}} \left[\int_{\mathbb{R}} |f(x-y) - f(x)|^p \, \varphi_\varepsilon(y) \, dy \right] dx$$

<u>Fubini</u>

$$= \int_{\mathbb{R}} \int_{\mathbb{R}} |f(x-y) - f(x)|^p \, \varphi_\varepsilon(y) \, dx \, dy$$

$$= \int_{\mathbb{R}} \varphi_\varepsilon(y) \left[\underbrace{\int_{\mathbb{R}} |f(x-y) - f(x)|^p \, dx}_{= h(y)} \right] dy$$

$$= \int_{|y| < \varepsilon} \varphi_\varepsilon(y) \, h(y) \, dy$$

$$\leq \sup_{|y| < \varepsilon} h(y) \int_{|y| < \varepsilon} \varphi_\varepsilon(y) \, dy$$

$$= \sup_{|y|<\varepsilon} h(y).$$

So we have

$$\int_{\mathbb{R}} |f^{\varepsilon}(x) - f(x)|^p \, dx$$

$$\leq \sup_{|y|<\varepsilon} \int_{\mathbb{R}} |f(x-y) - f(x)|^p \, dx$$

$$= \sup_{|y|<\varepsilon} \| \tau_y(f) - f \|_{L^p}^p$$

$\longrightarrow 0$ as $\varepsilon \to 0^+$ by earlier result.

(iv) let f be Bounded

& not continuous, say

$$f(x) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & 1 < x. \end{cases}$$

Then f^ε is smooth & <u>If</u>

$f^\varepsilon \to f$ in $L^\infty(\mathbb{R})$ then this

means $f^\varepsilon \to f$ uniformly in $\mathbb{R}$

$\Rightarrow f$ continuous,

a contradiction.

Question 2

(i) $0 = \Delta u = u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta}$

Try $u = R(r)\, g(\theta)$.

$$0 = R'' g + \frac{R' g}{r} + \frac{R g''}{r^2}$$

$$\frac{r^2 R''}{R} + \frac{r R'}{R} = -\frac{g''}{g} = \lambda$$

$$\begin{cases} g''(\theta) + \lambda g(\theta) = 0 & 0 < \theta < \alpha \\ g(0) = g(\alpha) = 0. \end{cases}$$

~~Only things~~ that gives us

Anything is $\lambda > 0$.

$$g(\theta) = A\sin(\sqrt{\lambda}\,\theta) + B\cos(\sqrt{\lambda}\,\theta)$$

$$g(0) = 0 \Rightarrow B = 0$$

$$\therefore g(\theta) = \sin(\sqrt{\lambda}\,\theta)$$

$$0 = g(\alpha) = \sin(\sqrt{\lambda}\,\alpha)$$

$$\therefore \sqrt{\lambda}\,\alpha = k\pi$$

$$\lambda_k = \left(\frac{k\pi}{\alpha}\right)^2 \quad \Big| \quad k \geq 1$$

$$r^2 R_k'' + r R_k' - \lambda_k R_k = 0 \qquad r > 0$$

Try $R_k(r) = r^{\beta}$

$$\beta^2 = \lambda_k = \left(\frac{k\pi}{\alpha}\right)^2$$

$$\beta_k = \pm \frac{k\pi}{\alpha}$$

[11]

$$u(r,\theta) = \sum_{k=1}^{\infty} \left(A_k r^{\frac{k\pi}{a}} + \frac{B_k}{r^{\frac{k\pi}{a}}} \right) \cancel{\sin(\ldots)} \sin\left(\frac{k\pi}{a}\theta\right)$$

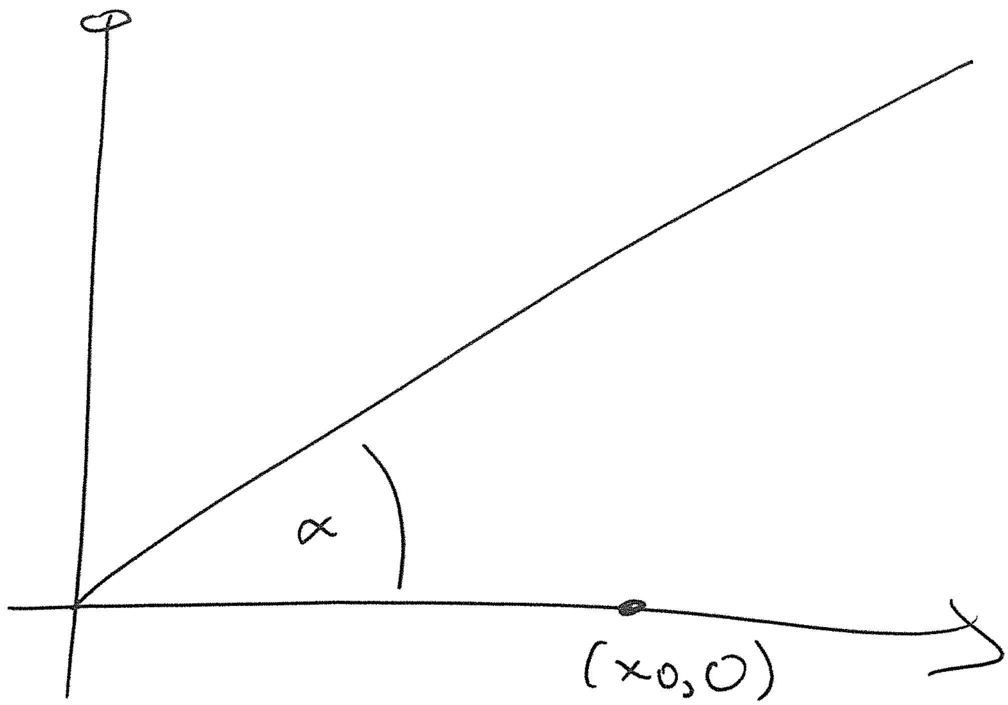
most general solution.

(ii) $\begin{cases} \Delta u = 0 & \Omega \\ \partial_r u = 0 & \partial\Omega. \end{cases}$

We want to justify that

$\partial_r u = 0$ on $\partial\Omega \setminus \{0\}$ same as

$u_\theta = 0$ on $\partial\Omega \setminus \{0\}$.



For $(x_0, 0)$, $x_0 > 0$ we see

$$-\partial_r u(x_0, 0) = +\partial_y u(x_0, 0).$$

$$= \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}.$$

$$\frac{\partial r}{\partial y} = \frac{y}{r} \qquad \text{so} \qquad \left.\frac{\partial r}{\partial y}\right|_{(x_0, 0)} = 0 \quad \xi$$

$$\tan\theta = \frac{y}{x} \qquad (\sec^2\theta)\frac{\partial \theta}{\partial y} = \frac{1}{x}$$

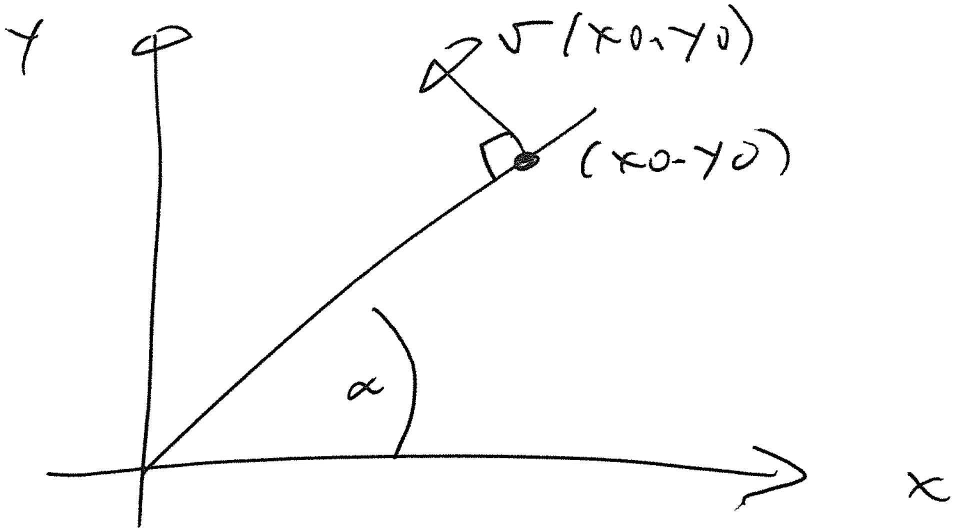
$$\text{at } (x_0, 0) \; \theta \Rightarrow \frac{\partial \theta}{\partial y} = \frac{1}{x_0} \neq 0$$

so at $(x_0, 0)$ we have

$$-\partial_v u(x_0, 0) = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial \theta}\frac{\partial \theta}{\partial y}$$

$$= \frac{1}{x_0}\frac{\partial u}{\partial \theta}$$

so $\partial_v u(x_0, 0) = 0$ iff $\left.\frac{\partial u}{\partial \theta}\right|_{(x_0, 0)} = 0.$



$$v(x_0, y_0) = \begin{pmatrix} -\sin(\alpha) \\ \cos(\alpha) \end{pmatrix}.$$

$$U_x = U_r\, r_x + U_\theta\, \theta_x$$

$$r_x = \frac{\partial r}{\partial x} \qquad \theta_x = \frac{\partial \theta}{\partial x}.$$

$$U_y = U_r\, r_y + U_\theta\, \theta_y$$

From Here on everything evaluated at $(x,y) = (x_0, y_0)$

so

$$d_r U(x_0, y_0)$$

$$= \frac{U_r}{r}\left[-x\sin(\alpha) + y\cos(\alpha)\right] + U_\theta\left[\theta_y\cos(\alpha) - \theta_x\sin(\alpha)\right]$$

$$= \frac{U_r}{r}\cdot 0 + U_\theta\left[\theta_y\cos(\alpha) - \theta_x\sin(\alpha)\right]$$

$$= U_\theta\left[\theta_y\cos(\alpha) - \theta_x\sin(\alpha)\right]$$

show this $\neq 0$

so $\partial_\nu u(x_0, y_0) = 0$ iff $u_\theta\Big|_{(x_0,y_0)} = 0$.

now do separation of variables to get cosine series.

(iii) Try $u(r,\theta) = r^t w(\theta)$.

$w \geq 0$

$$0 \overset{\text{want}}{=} \Delta u + u^p$$

$$= u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} + u^p$$

$$= t(t-1) r^{t-2} w(\theta)$$

$$+ t r^{t-2} w(\theta) + r^{t-2} w''(\theta)$$

$$+ r^{tp} w(\theta)^p \qquad 0 < r$$

$0 < \theta < \alpha$

want $tp = t - 2$

so $t(p-1) = -2$

Or $\boxed{t = \dfrac{-2}{p-1}}$.

Then w needs to satisfy

$$\begin{cases} -w''(\theta) = (w(\theta))^p + t^2 w(\theta) & 0 < \theta < \alpha \\ w(0) = w(\alpha) = 0 \\ w(\theta) > 0 \text{ in } (0, \alpha) \end{cases}$$

Since we want $u > 0$ in Ω.

you can't solve at this point this B.V.P. for w.

Question 3

(i) put $c(x) = (u(x))^2$ then

u satisfies

$$\begin{cases} -\Delta u + c(x) u = f \geq 0 & \Omega \\ u = 0 & \partial\Omega \end{cases}$$

So by max principle $u \geq 0$ in Ω.

(ii) let $x_0 \in \Omega$ s.t.

$u(x_0) = \max_{\Omega}(u)$. Then

$-\Delta u(x_0) \geq 0$

$\Rightarrow (u(x_0))^3 \leq f(x_0)$

$\leq \sup_{\Omega}(f)$

$$\Rightarrow \quad \sup_{\sim} |u(x)| \leq \left(\sup_{\sim} |f(x)| \right)^{\frac{1}{3}}.$$

#4 $\quad g'(t) \leq \beta(t) g(t) \qquad t \in (0,T)$

write as $g'(t) - \beta(t) g(t) \leq 0$

$$\mu(t) = e^{\int_0^t (-\beta(\tau)) d\tau} \quad \text{integrating factor.}$$

$$\frac{d}{dt}\left[\mu(t)\, g(t)\right] \leq 0$$

$$\Rightarrow \quad \mu(t) g(t) - \mu(0) g(0) \leq 0 \qquad 0 < t < T$$

$$g(t) \leq \frac{g(0)}{\mu(t)} = g(0)\, e^{\int_0^t \beta(\tau) d\tau}.$$

#5

(i) $m(t) := \int_0^{\pi} u(x,t)\,dx$

$$m'(t) = \int_0^{\pi} u_t(x,t)\,dx$$

$$= \int_0^{\pi} u_{xx}(x,t)\,dx$$

$$= u_x(x,t)\Big|_{x=0}^{x=\pi}$$

$$= u_x(\pi,t) - u_x(0,t)$$

$$= 0 - 0 = 0 \quad \text{for } t > 0.$$

so $m(t) = m(0) = \int_0^{\pi} \phi(x)\,dx$

$$= 0.$$

(iii) $g(t) = \int_0^\pi (u(x,t))^2 dx$

$$g'(t) = 2\int_0^\pi u\, u_t\, dx$$

$$= 2\int_0^\pi u\, u_{xx}\, dx$$

$$= 2\left[-\int_0^\pi u_x^2\, dx + u\, u_x \Big|_{x=0}^{x=\pi}\right]$$

$$= (-2)\int_0^\pi (u_x(x,t))^2 dx$$

$$\leq -2\int_0^\pi (u(x,t))^2 dx \quad \text{By the inequality}$$

$$= -2g(t)$$

so $g'(t) + 2g(t) = 0$ for $t > 0$

$\mu(t) = e^{2t}$
integrating factor

$$\frac{d}{dt}\left[e^{2t} g(t)\right] = 0$$

$$e^{2t} g(t) = g(0)$$

or $g(t) = e^{-2t} g(0)$

or

$$\boxed{\int_0^\pi (u(x,t))^2\,dx = e^{-2t} \int_0^\pi (\phi(x))^2\,dx}$$