Solutions

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#1

let u, v solve (1)

$w(x) = u(x) - v(x)$. Then

$$\begin{cases} -\Delta w + c(x) w = 0 & \Omega \\ w = 0 & \partial\Omega. \end{cases}$$

multiply by w & integrate by parts

$$\underbrace{\int_\Omega |\nabla w|^2 dx}_{\geq 0} + \underbrace{\int_\Omega c(x) w^2 dx}_{\geq 0} = 0$$

$\Rightarrow \int_\Omega |\nabla w|^2 dx = 0$. In class we showed if w nice & $w = 0$ on $\partial\Omega$ then $\Rightarrow w \equiv 0$.

Question 2

Let U_i solve (3)

$w(x) = U_1(x) - U_2(x)$.

Then

$$-\Delta w(x) + h(U_1(x)) - h(U_2(x)) = 0 \text{ in}$$

$w = 0$ on $\partial\Omega$.

$$\text{set } c(x) := \begin{cases} \dfrac{h(U_1(x)) - h(U_2(x))}{U_1(x) - U_2(x)} & U_1(x) \neq U_2(x) \\ 0 & \text{otherwise.} \end{cases}$$

~~A = \{x \in \Omega : U_1(x) \neq U_2(x)\}.~~

$$\begin{cases} -\Delta w(x) + C(x)\, w(x) = 0 & \Omega \\ w = 0 & \partial\Omega. \end{cases}$$

Claim $C(x) \geq 0$.

If $C(x) = 0$ done; take $U_1(x) \neq V_2(x)$. Then by MVT $\exists\, \gamma(x)$ between $U_1(x)$ & $V_2(x)$

$$\frac{h(U_1(x)) - h(V_2(x))}{U_1(x) - V_2(x)} = h'(\gamma(x)) \geq 0.$$

To directly apply question I ~~ne~~ need $0 \leq C(x) \leq M$.

~~Hence~~ let $R > 0$ s.t.

$$|U_i(x)| \le R \qquad i = 1,2$$
$$x \in \bar{\Omega}.$$

Then

$$c(x) = h'(\gamma(x))$$

$$\le \sup_{|z| \le R} h'(z)$$

But h smooth so this

is bounded.

Now Apply Question 1.

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Question 3

$u(x,y) = e^{\alpha x} \sin(\beta y).$

B.C. $u(x,0) = 0$ Fine for Any β.

$0 = u(x,\pi) = e^{\alpha x} \sin(\beta \pi)$

want $\beta = k \geq 1$ integer.

PDE

$0 = u_{xx} + u_{yy}$

$= \alpha^2 e^{\alpha x} \sin(\beta y)$

$+ e^{\alpha x} (-\beta^2) \sin(\beta y)$

$= (\alpha^2 - \beta^2) e^{\wedge} \sin(\beta y).$

take $\beta = \kappa = 1$

need $\alpha^2 = \beta^2 = 1$

so take $\alpha = 1$

so $u(x,y) = e^x \sin(y)$.

$\boxed{\text{Question 4}}$

$$M(t) = \int_\Omega u(x,t)\,dx$$

$$M'(t) = \frac{d}{dt}\left[\int_\Omega u(x,t)\,dx\right]$$

$$= \int_\Omega u_t(x,t)\,dx$$

$$= \int_\Omega \Delta u(x,t)\,dx \qquad \text{by PDE}$$

$$= \int_{\partial\Omega} \underbrace{\partial_\nu u}_{\to 0}\, dS(x) \qquad \text{by Green's formula}$$

$$= 0.$$

Question 5

$$E(t) = \frac{1}{2}\int_\Omega u_t^2\,dx + \frac{1}{2}\int_\Omega |\nabla u|^2\,dx - \frac{1}{4}\int_\Omega u^4\,dx$$

$$E'(t) = \int_\Omega u_t u_{tt}\,dx + \int_\Omega \nabla u \cdot \nabla(u_t)\,dx - \int_\Omega u^3 u_t\,dx$$

$$= \int_\Omega u_t u_{tt}\,dx + \int_\Omega (-\Delta u) u_t\,dx + \int_{\partial\Omega} (\partial_\nu u)\, \underbrace{u_t}_{=0}\,dx - \int_\Omega u^3 u_t\,dx$$

since $u=0$ on $\partial\Omega$.

$$= \int_\Omega u_t \, [\underbrace{u_{tt} - \Delta u - u^3}_{\overset{\parallel}{0} \text{ By PDE}}] \, dx$$

$$= 0.$$