Math7460 Homework 2

October 7, 2019

Question 1 is optional. If you do Question 1 then you can skip one of the other questions.

Question 1. (Analysis question) Let $0 \leq \phi$ be smooth and suppose its support is is $[-1, 1]$ and $\int_{\mathbb{R}} \phi(x)dx = 1$. For $\varepsilon > 0$ small we set $\phi_\varepsilon(x) := \varepsilon^{-1}\phi(\frac{x}{\varepsilon})$. Given a function f we define the 'mullification of f by

$$f^\varepsilon(x) := (\phi_\varepsilon * f)(x) = (f * \phi_\varepsilon)(x)$$

where

$$(\phi_\varepsilon * f)(x) := \int_{\mathbb{R}} \phi_\varepsilon(y) f(x - y)dy.$$

The goal of this question is to show that $f^\varepsilon \to f$ in some sense...depending on f. Recall in class we use the above formula's to half proof that f^ε is smooth. One comment to keep in mind is, depending on what you are doing you might want to consider $\phi_\varepsilon * f$ or $f * \phi_\varepsilon$. Also you might need to use some facts about ϕ_ε (its integral and its support).

(i) Suppose $f \in C(\overline{\mathbb{R}})$ (by this I mean f is uniformly continuous on $\mathbb{R}$). Show $f^\varepsilon \to f$ uniformly on $\mathbb{R}$ (you might want to look at how to write $f^\varepsilon - f$ from part (iii)).

(ii) Let $1 \leq p < \infty$. Given $y \in \mathbb{R}$ define the 'shift' operator by $(T_y f)(x) := f(x - y)$. Our goal is to show that given $f \in L^p(\mathbb{R})$ that $\|T_y f - f\|_{L^p} \to 0$. One can do this directly with some measure theory (and you can do that if you know the required stuff... probably anything like this follows directly by "Lebesgue differentiation theorem" or maybe "Lebesgue Density point theorem". Instead we will use a density result (it appears we might be using a circular argument if we assume a density result... but we can prove this density result using some other results from analysis... including 'Stone-Weirstrass') (above is just prelude to question...you don't need to write anything for the above).
Suppose $C_c(\mathbb{R})$ (continuous compactly supported) is dense in $L^p(\mathbb{R})$. So let $f \in L^p(\mathbb{R})$ and let $\delta > 0$ small; then there exists $g \in C_c(\mathbb{R})$ such that $\|f - g\|_{L^p} < \delta$. Then we have

$$T_y f - f = (T_y f - T_y g) + (T_y g - g) + (g - f)$$

and hence

$$\|T_y f - f\|_{L^p} \leq \|T_y f - T_y g\|_{L^p} + \|T_y g - g\|_{L^p} + \|g - f\|_{L^p}.$$

SHOW $\|T_y f - T_y g\|_{L^p} \leq \delta$. We'd then have

$$\|T_y f - f\|_{L^p} \leq 2\delta + \|T_y g - g\|_{L^p}.$$

Now **SHOW** $\limsup_{y\to 0} \|T_y g - g\|_{L^p} = 0$. Now **SHOW** from this we have $\lim_{y\to 0} \|T_y f - f\|_{L^p} = 0$.

(iii) Let $1 \leq p < \infty$ and set f^ε as above. **SHOW** we can write

$$f^\varepsilon(x) - f(x) = \int_{\mathbb{R}} \phi_\varepsilon(y)(f(x - y) - f(x))dy,$$

and hence we have

$$|f^\varepsilon(x) - f(x)| \le \int_{\mathbb{R}} |f(x-y) - f(x)| \phi_\varepsilon(y) dy.$$

We now want to raise both sides to the power p. One way to do this is to use **Jensen's inequality**.

Jensen inequality. Suppose μ is a probability measure on $\mathbb{R}$; ie. $\mu(\mathbb{R}) = 1$. Then given any convex function $H : \mathbb{R} \to \mathbb{R}$ one has

$$H(\int_{\mathbb{R}} g(x) d\mu(x)) \le \int_{\mathbb{R}} H(g(x)) d\mu(x)$$

for all functions g.

Returning to our problem; we see the convex function we want to use is $H(t) = |t|^p$. **SHOW** you can apply Jensen's inequality (you will need to indicate what exactly your probability measure is) to get

$$|f^\varepsilon(x) - f(x)|^p \le \int_{\mathbb{R}} |f(x-y) - f(x)|^p \; \phi_\varepsilon(y) dy.$$

Now integrate in x and **SHOW** we can arrive at

$$\int_{\mathbb{R}} |f^\varepsilon(x) - f(x)|^p dx \le \sup_{|y|<\varepsilon} \int_{\mathbb{R}} |f(x-y) - f(x)|^p dx.$$

Using this **SHOW** $\int_{\mathbb{R}} |f^\varepsilon(x) - f(x)|^p dx \to 0$ as $\varepsilon \searrow 0$.

(iv) Here we show one cannot have the above result for $p = \infty$. **SHOW** given $f \in L^\infty$ we CANNOT expect to have $f^\varepsilon \to f$ in L^∞. Hint. f^ε is smooth and what kind of convergence is L^∞.

Question 2. Let $0 < \alpha < \frac{\pi}{2}$ and set $\Omega := \{x \in \mathbb{R}^2 : r > 0, \theta \in (0, \alpha)\}$ (we are using polar co-ordinates). **EDIT for this question.** By a solution I mean u is smooth in Ω but maybe its badly behaved at the vertex.

(i) Using seperation of variables find the most general solution of

$$\Delta u = 0 \quad \text{in } \Omega, \qquad u = 0 \quad \text{on } \partial\Omega \backslash \{0\}.$$

(ii) Using seperation of variables find the most general solution of

$$\Delta u = 0 \quad \text{in } \Omega, \qquad \partial_\nu u = 0 \quad \text{on } \partial\Omega \backslash \{0\}.$$

(iii) Let $1 < p < \infty$. Look for a positive solution of $-\Delta u = u^p$ in Ω with $u = 0$ on $\partial\Omega \backslash \{0\}$. Hint. Look for solutions of the form $u(r, \theta) = g(r) w(\theta)$. (you should be able to figure out exactly what $g(r)$ is and then you will have some boundary value problem you will need satisfied by $w(\theta)$ on $\theta \in (0, \alpha)$ and this you won't be able to solve at this point).

Question 3. Consider

$$\begin{cases} -\Delta u + u^3 &= f(x) \qquad \text{in } \Omega, \\ u &= 0 \qquad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where $f \geq 0$ is bounded and smooth and Ω is bounded.

(i) Recall we have a maximum principle that says: if w satisfies $-\Delta w(x) + C(x)w = g \geq 0$ in Ω with $w = 0$ on $\partial\Omega$ and $C \geq 0$ then $w \geq 0$ in Ω. Using this show that a solution of (1) is nonnegative.

(ii) Directly using the equation show that

$$\sup_{\Omega} |u(x)| \leq \left(\sup_{\Omega} |f(x)| \right)^{\frac{1}{3}}.$$

(Remember u is nice...so attains its sup at some $x_0 \in \Omega$)

Question 4. Suppose $0 < T < \infty$ and $g \in C[0,T) \cap C^1(0,T)$. Suppose we have

$$g'(t) \leq \beta(t) g(t) \qquad t \in (0,T),$$

and we assume β is bounded on $[0,T)$ (maybe blows up at the endpoint). Show

$$g(t) \leq g(0) e^{\int_0^t \beta(\tau) d\tau}, \qquad \forall t \in (0,T).$$

Hint. Note this is NOT an ode. Think about 'integrating factor' method.

Question 5. Suppose $u = u(x,t)$ a smooth solution of

$$\begin{cases} u_t &= u_{xx} \qquad (x,t) \in (0,\pi) \times (0,\infty), \\ u_x(0,t) &= u_x(\pi,t) = 0 \qquad t > 0, \\ u(x,0) &= \phi(x) \qquad x \in (0,\pi). \end{cases} \tag{2}$$

We assume $\int_0^\pi \phi(x) dx = 0$.

(i) Show for each $t > 0$ we have $\int_0^\pi u(x,t) dx = 0$. (Work directly with the pde and not using Fourier series)

(ii) You can assume there the following holds: for all $\psi = \psi(x)$ nice with $\int_0^\pi \psi(x) dx = 0$ one has

$$\int_0^\pi (\psi'(x))^2 dx \geq \int_0^\pi (\psi(x))^2 dx.$$

Set $g(t) := \int_0^\pi (u(x,t))^2 dx$. Show $g(t) \to 0$ as $t \to \infty$.
Hint. Start with g and take a derivative and then play around with the approach from Question 4.