Homework 1; Due September 25

September 19, 2015

1 Mandatory questions

Question 1. Consider the equation

$$\begin{cases} -\Delta u + C(x)u &= f(x) \quad \text{in } \Omega, \\ u &= g(x) \quad \text{on } \partial\Omega, \end{cases} \tag{1}$$

where Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary and where there is some $M < \infty$ such that $C(x)$ satisfies $0 \le C(x) \le M$ in Ω. Show there is at most one solution of (1).

An example where uniqueness fails. The above result can be weakened to: "provided $C(x)$ is not too negative then (1) has at most one solution. We now give an example where $C(x)$ is too negative and one loses uniqueness. Consider $u(x) := \sin(3x)$. Then note that

$$-u''(x) = 9u(x) \quad x \in (0, \pi), \qquad u(0) = u(\pi) = 0.$$

So re-arranging this we see that u_k is a non-zero solution of

$$-u''(x) - 9u(x) = 0 \quad x \in (0, \pi), \qquad u(0) = u(\pi) = 0. \tag{2}$$

But of course $u = 0$ is also a solution of (2).

Question 2. Consider

$$\begin{cases} -\Delta u + h(u) &= f(x) \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{3}$$

where Ω a bounded domain in $\mathbb{R}^N$ with smooth boundary and where $h : \mathbb{R} \to \mathbb{R}$ is smooth with $h'(z) \ge 0$ for all $z \in \mathbb{R}$. Show (3) has at most one solution.

Hint. Start as with a linear problem: let u, v be solutions and then define $w(x) := u(x) - v(x)$. What equation does w solve? You will need to use Question 1 with a particular $C(x)$ defined in terms of h, u and v.

Question 3. Here we show that if one removes the assumption of Ω bounded then one can lose uniqueness of solutions. Consider

$$\begin{cases} u_{xx}(x,y) + u_{yy}(x,y) = 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4}$$

where $\Omega := \{(x,y) : x \in \mathbb{R}, 0 < y < \pi\}$. Of course $u = 0$ is a solution of (4). Find another solution of the form $u(x,y) = e^{\alpha x}\sin(\beta y)$ where $\alpha, \beta \in \mathbb{R}$ are to be determined.

Question 4. In this question we use the maximum principle to obtains bounds on a solution. Suppose u is a smooth solution of

$$\begin{cases} -\Delta u = f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\Omega \subset B_R := \{x \in \mathbb{R}^N : |x| < \mathbb{R}\}$ and where $0 \leq f(x) \leq M$. Define $v(x) := \frac{M}{2N}(R^2 - |x|^2)$. Using the maximum principle show that

$$\sup_\Omega u \leq \frac{MR^2}{2N}.$$

Note that when the "diameter of Ω" gets big, ie $\sup\{|x-y| : x, y \in \Omega\}$ then this estimate gets worse and worse (Note the R^2). Below we show an approach to get good estimates that really only depend on the set Ω not being big in one direction.

A result better than Question 4. (This is not a question) Suppose $0 \leq v$ is a solution of

$$\begin{cases} -\Delta v + v = g(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{6}$$

where $0 \leq g$. Note that $0 \leq v$ by assumption and so the maximum is attained in Ω. Suppose $\sup_\Omega v = v(x_0)$ for $x_0 \in \Omega$. Then we have (using the single variable calculus trick) that

$\Delta v(x_0) \le 0$ and so $-\Delta v(x_0) \ge 0$. From this note that

$$v(x_0) \le -\Delta v(x_0) + v(x_0) = g(x_0) \le \sup_\Omega g$$

and hence we have

$$\sup_\Omega v \le \sup_\Omega g.$$

We now return to the question. Suppose Ω is a bounded domain in $\mathbb{R}^N$ and suppose that $\Omega \subset \{x \in \mathbb{R}^N : |x_1| < \frac{\pi}{3}\}$. Suppose u solves (5) where $0 \le f$ and so $0 \le u$. Consider the change of variables:

$$u(x) = E(x)v(x),$$

where $E(x) > 0$ in $\overline{\Omega}$, is to be determined later. Then note that

$$-f(x) = \Delta u(x) = E(\Delta v) + 2\nabla E \cdot \nabla v + v\Delta E.$$

Also note that $v = 0$ on $\partial\Omega$. Since E is positive we can divide by E. So v solves

$$\begin{cases} -\Delta v - \frac{2\nabla E}{E} \cdot \nabla v + \frac{(-\Delta E)}{E} v = \frac{f}{E} & \text{in } \Omega, \\ v = 0 & \text{on } \partial\Omega. \end{cases} \tag{7}$$

We would now like to pick E such that this equation resembles (6). So let try and pick E such that $\frac{-\Delta E}{E} = 1$. Pick $E(x) = \cos(x_1)$. Then note that $E(x) = \cos(x_1) \ge \frac{1}{2}$ in Ω. Also note that $-\Delta E(x) = -\frac{d^2}{dx^2}\cos(x) = \cos(x) = E(x)$. Now $0 \le v$ and we would like to get upper bound on v. So let $x_0 \in \Omega$ be such that $v(x_0) = \sup_\Omega v$ and so $-\Delta v(x_0) \ge 0$ and $\nabla v(x_0) = 0$. So plugging into the pde we have

$$v(x_0) \le \left(-\Delta v - \frac{2\nabla E}{E} \cdot \nabla v + \frac{(-\Delta E)}{E} v \right)\Big|_{x_0} = \frac{f(x_0)}{E(x_0)}.$$

So we have

$$\sup_\Omega \frac{u(x)}{E(x)} \le \frac{f(x_0)}{E(x_0)} \le \sup_\Omega \frac{f}{E}$$

but we have

$$\frac{\sup_\Omega u}{\sup_\Omega E} \le \sup_\Omega \frac{u(x)}{E(x)}$$

and

$$\sup_\Omega \frac{f}{E} \le \frac{\sup_\Omega f}{\inf_\Omega E}$$

and hence we get

$$\frac{\sup_\Omega u}{\sup_\Omega E} \le \frac{\sup_\Omega f}{\inf_\Omega E}$$

which gives

$$\sup_\Omega u \le \frac{\sup_\Omega E}{\inf_\Omega E} \sup_\Omega f.$$

Now with our choice of E we have $\sup_\Omega E = 1$ and $\inf_\Omega E \geq \frac{1}{2}$. So we have

$$\sup_\Omega u \leq 2 \sup_\Omega f.$$

There is nothing special about $\frac{\pi}{3}$. For other widths of domains try $E(x) = \cos(\lambda x_1)$ where you pick $\lambda > 0$ appropriately.

2 Extra questions for graduate students

Question 100; Mandatory question for graduate students. (L^2 convergence of the cosine series) Goal of the this problem is to show that given $f \in L^2(0, \pi)$ that the cosine series converges in L^2 to f. We begin with some definitions.

Consider K a compact topological space and let $C(K)$ denote the set of real valued continuous functions on K. We say $\mathcal{A} \subset C(K)$ is a **subalgebra** provided its a linear space over $\mathbb{R}$ and provided $f, g \in \mathcal{A}$ then $fg \in \mathcal{A}$ (here $(fg)(x) := f(x)g(x)$). We say $\mathcal{A}$ seperates K provided for all $x, y \in K$ with $x \neq y$ there is some $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Theorem 1. *(Stone-Weistrass)*
Let $K, C(K)$ be as above and suppose $\mathcal{A} \subset C(K)$ is a subalgebra with seperates K. In addition suppose $1 \in \mathcal{A}$. Then $\mathcal{A}$ is dense in $C(K)$.

Consider

$$\mathcal{A} := \{f : \quad \text{where } f(x) = \sum_{k=0}^{\infty} c_k \cos(kx) \text{where } c_k \in \mathbb{R} \text{ and all but a finite number are zero}\}.$$

(I) Using $\cos(a+b) = \cos(a)\cos(b) - \sin(a)\sin(b)$ and $\cos(a-b) = \cos(a)\cos(b) + \sin(a)\sin(b)$, show that $\mathcal{A}$ is a subalgebra on $C[0, \pi]$ which seperates $[0, \pi]$. Using the Stone-Weirstrass theorem conclude that $\mathcal{A}$ is dense in $C[0, \pi]$. Define

$$\mathcal{A}_n := \{f \in \mathcal{A} : c_k = 0 \text{ for all } k \geq n+1\}.$$

Conclude that given $f \in C[0, \pi]$ that

$$\inf_{g \in \mathcal{A}_n} \sup_{[0,\pi]} |g(x) - f(x)| \to 0$$

as $n \to \infty$.
(II) Given $n \geq 2$ let $S_n(f)$ denote the n^{th} partial cosine series: so

$$S_n(f)(x) := \sum_{k=0}^{n} a_k \cos(kx)$$

where

$$a_k = \frac{2}{\pi} \int_0^\pi f(x) \cos(kx) dx, \qquad a_0 = \frac{1}{\pi} \int_0^\pi f(x) dx.$$

Our goal of this part is to show that

$$\|f - S_n(f)\|_{L^2(0,\pi)} = \inf_{g \in \mathcal{A}_n} \|f - g\|_{L^2(0,\pi)}.$$

First note that $S_n(f) \in \mathcal{A}_n$ and hence we have $\|f - S_n(f)\|_{L^2(0,\pi)} \geq \inf_{g \in \mathcal{A}_n} \|f - g\|_{L^2(0,\pi)}$ and so we just need to show the other direction. Let $g \in \mathcal{A}_n$ and assume

$$g(x) = \sum_{k=0}^{n} c_k \cos(kx)$$

and lets compute $G(c) := \|g - f\|^2_{L^2(0,\pi)}$; we are viewing $G : \mathbb{R}^{n+1} \to \mathbb{R}$. Show that (use can accept the orthogonality conditions for $\cos(kx)$ and be careful with $k = 0$ term) that

$$\frac{G(c)}{\pi} = c_0^2 - 2c_0 a_0 + \frac{1}{2} \sum_{k=1}^{n} c_k^2 - \sum_{k=1}^{n} c_k a_k + \frac{1}{\pi} \int_0^{\pi} f(x)^2 dx.$$

We would now like to minimize G over $\mathbb{R}^{n+1}$. Show that G has a global minimum at $c = a$ where $a = (a_0, a_1, ..., a_n)$ are defined as above. What can you conclude.

(III) We now tie the different parts of the question together. Fix $f \in L^2(0,\pi)$ and let $0 < \varepsilon$ be small. Then by some density results there is some $h \in C[0,\pi]$ such that $\|f - h\|_{L^2} < \varepsilon$. Then by part (I) and (II) we have

$$\|S_n(h) - h\|_{L^2} \leq \inf_{g \in \mathcal{A}_n} \|g - h\|_{L^2} \leq \inf_{g \in \mathcal{A}_n} \sup_{[0,\pi]} |g(x) - h(x)| \sqrt{\pi} \to 0$$

as $n \to \infty$. Then note that we have

$$\|S_n(f) - f\|_{L^2} \leq \|S_n(f) - S_n(h)\|_{L^2} + \|S_n(h) - h\|_{L^2} + \|h - f\|_{L^2},$$

and recall we have $\|h - f\|_{L^2} < \varepsilon$. Also by Parseval we have $\|S_n(f) - S_n(h)\|_{L^2} = \|S_n(f - h)\|_{L^2} \leq C\|f - h\|_{L^2} \leq C\varepsilon$ where C independent of ε, f, h, n. So we end up with something like

$$\|S_n(f) - f\|_{L^2} \leq (C + 1)\varepsilon + \|S_n(h) - h\|_{L^2}$$

and so

$$\limsup_n \|S_n(f) - f\|_{L^2} \leq (C + 1)\varepsilon$$

and since $\varepsilon > 0$ is arbitrary we have $\|S_n(f) - f\|_{L^2} \to 0$.