A new variational principle, convexity and supercritical Neumann problems *

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Abstract

Utilizing a new variational principle that allows dealing with problems beyond the usual locally compactness structure, we study problems with a supercritical nonlinearity of the type

$$\begin{cases} -\Delta u + u = a(x) f(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

To be more precise, Ω is a bounded domain in $\mathbb{R}^N$ which satisfies certain symmetry assumptions; Ω is a domain of 'm revolution' ($1 \le m < N$ and the case of $m = 1$ corresponds to radial domains) and where $a > 0$ satisfies compatible symmetry assumptions along with monotonicity conditions. We find positive nontrivial solutions of (1) in the case of suitable supercritical nonlinearities f by finding critical points of I where

$$I(u) = \int_\Omega \left\{ a(x) F^* \left(\frac{-\Delta u + u}{a(x)} \right) - a(x) F(u) \right\} dx,$$

over the closed convex cone K_m of nonnegative, symmetric and monotonic functions in $H^1(\Omega)$ where $F' = f$ and where F^* is the Fenchel dual of F. We mention two important comments: firstly that there is a hidden symmetry in the functional I due to the presence of a convex function and its Fenchel dual that makes it ideal to deal with super-critical problems lacking the necessary compactness requirement. Secondly the energy I is not at all related to the classical Euler-Lagrange energy associated with (1). After we have proven the existence of critical points u of I on K_m we then unitize a new abstract variational approach (developed by one of the present authors in [27, 29]) to show these critical points in fact satisfy $-\Delta u + u = a(x) f(u)$.

In the particular case of $f(u) = |u|^{p-2} u$ we show the existence of positive nontrivial solutions beyond the usual Sobolev critical exponent.

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1 Introduction

In this paper we consider the existence of positive solutions of the Neumann problem given by

$$\begin{cases} -\Delta u + u = a(x) f(u), & \text{in } \Omega, \\ u > 0, & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Ω is a bounded domain in $\mathbb{R}^N$ which satisfies certain symmetry assumption and where a is a positive sufficiently smooth function which also has some symmetry and monotonicity properties. When f is a subcritical nonlinearity one can utilize a standard variational approach to obtain solutions of (2). With this in mind our interest is in the case of f a supercritical nonlinearity; for example $f(u) = |u|^{p-2}u$ where $p > 2^* := \frac{2N}{N-2}$. Our approach will be to use a new variational approach, see Theorem 1.8 (developed in [29, 27]) over a class of functions with certain monotonicity properties, to obtain a positive solution of (2). The extra monotonicity of the functions will give us increased ranges on the Sobolev imbeddings and this allows one to handle suitable supercritical nonlinearities.

1.1 Main results and symmetry assumptions on Ω

The domains we consider are 'domains of m revolution' (which we define precisely below) and of course the most basic case is a radial domain. The next level would be what are called domains of double revolution. Our motivation to study these special domains stems from [9] where they considered domains of double revolution in the context of the regularity of the extremal solution associated to nonlinear eigenvalue problems of the form $-\Delta u = \lambda f(u)$ in Ω with $u = 0$ on $\partial\Omega$. We now describe these domains.

Domains of double revolution. Consider writing $\mathbb{R}^N = \mathbb{R}^m \times \mathbb{R}^n$ where $m + n = N$ and $m, n \geq 1$. We define the variables s and t by

$$s^2 := x_1^2 + \cdots + x_m^2, \qquad t^2 := x_{m+1}^2 + \cdots + x_N^2.$$

We say that $\Omega \subset \mathbb{R}^N$ is a *domain of double revolution* if it is invariant under rotations of the first m variables and also under rotations of the last n variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^N : (s,t) \in U\}$ where U is a domain in $\mathbb{R}^2$ symmetric with respect to the two coordinate axes. In fact,

$$U = \left\{(s,t) \in \mathbb{R}^2 : x = (x_1 = s, x_2 = 0, ..., x_m = 0, x_{m+1} = t, ..., x_n = 0) \in \Omega\right\},$$

is the intersection of Ω with the (x_1, x_{m+1}) plane. Note that U is smooth if and only if Ω is smooth. We denote $\tilde{\Omega}$ to be the intersection of U with the first quadrant of $\mathbb{R}^2$. Note that given any function v defined in Ω, that depends only on the radial variables s and t, one has

$$\int_\Omega v(x)dx = c(m,n) \int_{\tilde{\Omega}} v(s,t) s^{m-1} t^{n-1} ds dt,$$

where $c(m,n)$ is a positive constant depending on n and m. Note that strictly speaking we are abusing notation here by using the same name; and we will continuously do this in this article. Given a function v defined on Ω we will write $v = v(s, t)$ to indicate that the function has this symmetry. We remark that generally one requires that $m, n \geq 2$, but in the current work we allow the case of m or n equal to 1 as well.

Example 1.1. *Let Ω be the cylinder $x^2+y^2 < 1$ with $-1 < z < 1$ in $\mathbb{R}^3$. Then Ω is a domain of double revolution. In fact, by letting $s^2 = x^2 + y^2$ and $t^2 = z^2$ one has that*

$$\Omega = \{(x, y, z) \in \mathbb{R}^3;\ |s|, |t| < 1\}.$$

Domains of triple revolution. For domains of tripe revolutions and higher we adopt a more uniform notation. Consider writing $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \mathbb{R}^{n_3}$ where $n_1 + n_2 + n_3 = N$ and define the variables t_i via

$$t_1^2 := x_1^2 + \cdots + x_{n_1}^2, \qquad t_2^2 := x_{n_1+1}^2 + \cdots + x_{n_1+n_2}^2, \qquad t_3^2 := x_{n_1+n_2+1}^2 + \cdots + x_N^2.$$

We say that $\Omega \subset \mathbb{R}^N$ is a *domain of triple revolution* if it is invariant under rotations of the first n_1 variables and also under rotations of the middle n_2 variables and the last n_3 variables. Equivalently, Ω is of the form $\Omega = \{x \in \mathbb{R}^N : (t_1, t_2, t_3) \in U\}$ where U is a domain in $\mathbb{R}^3$ symmetric with respect to the three coordinate axes. In fact,

$$U = \Big\{(t_1, t_2, t_3) \in \mathbb{R}^3; x = (x_1, ..., x_N) \in \Omega,\ \text{where } x_1 = t_1, x_{n_1+1} = t_2, x_{n_1+n_2+1} = t_3 \text{ and } x_i = 0 \text{ for } i \neq 1, n_1 + 1, n_1 + n_2 + 1\Big\},$$

is the intersection of Ω with the $(x_1, x_{n_1+1}, x_{n_1+n_2+1})$ plane. We denote $\tilde{\Omega}$ to be the intersection of U with the first "sector" of $\mathbb{R}^3$. Note that given any function v defined in Ω, that depends only on the radial variables t_1, t_2, t_3 one has

$$\int_\Omega v(x)dx = c \int_{\tilde{\Omega}} v(t_1, t_2, t_3) t_1^{n_1-1} t_2^{n_2-1} t_3^{n_3-1} dt_1 dt_2 dt_3.$$

for some constant $c = c(n_1, n_2, n_3)$.

Domains of m revolution. Consider writing $\mathbb{R}^N = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \times \cdots \times \mathbb{R}^{n_m}$ where $n_1 + \cdots + n_m = N$ and $n_1, ..., n_m \geq 1$. We say that $\Omega \subset \mathbb{R}^N$ is a *domain of m revolution* if it is invariant under rotations of the first n_1 variables, the next n_2 variables, ..., and finally in the last n_m variables. We define the variables t_i via

$$t_1^2 := x_1^2 + \cdots + x_{n_1}^2, \quad t_2^2 := x_{n_1+1}^2 + \cdots + x_{n_1+n_2}^2,$$

and similar for t_i for $3 \leq i < m$. Finally we define

$$t_m^2 := \sum_{k=n_1+n_2+\cdots+n_{m-1}+1}^{N} x_k^2.$$

We now define

$$U = \Big\{t \in \mathbb{R}^m; x = (x_1, ..., x_N) \in \Omega,\ \text{where } x_1 = t_1,\ x_{n_1+n_2+\cdots+n_{k-1}+1} = t_k \text{ for } 2 \leq k \leq m, \text{ and } x_i = 0 \text{ for } i \neq 1, n_1 + 1, n_1 + n_2 + 1, ..., n_1 + n_2 + ... + n_{m-1} + 1\Big\}.$$

We define $\tilde{\Omega} \subset \mathbb{R}^m$ to be the intersection of U with the first sector of $\mathbb{R}^m$. We now define the appropriate measure

$$d\mu_m(t) = d\mu_m^{(n_1,...,n_m)}(t_1,...,t_m) = \prod_{k=1}^{m} t_k^{n_k-1} dt_k.$$

Given any function v defined in Ω, that depends only on the radial variables $t_1, t_2, .., t_m$ one has

$$\int_\Omega v(x)dx = c(n_1,...,n_m) \int_{\tilde{\Omega}} v(t)d\mu_m(t),$$

where $c(n_1,...,n_m)$ just depends on $n_1,...,n_m$. Given that $\Omega \subset \mathbb{R}^N$ is a domain of m revolution with $\sum_{i=1}^m n_i = N$, let

$$G := O(n_1) \times O(n_2) \times ... \times O(n_m),$$

where $O(n_i)$ is the orthogonal group in $\mathbb{R}^{n_i}$ and consider

$$H_G^1 := \left\{ u \in H^1(\Omega) : gu = u \quad \forall g \in G \right\},$$

where $gu(x) := u(g^{-1}x)$. If $u \in H_G^1$, then u has symmetry compatibility with Ω, ie. $u(x)$ depends on just $t_1,...,t_m$ and we write this as $u(x) = u(t_1,...,t_m)$ where $(t_1,...,t_m) \in \tilde{\Omega}$.

Remark 1.2. *Now that we have clarified what we mean by a domain of m revolution we can further explain our results. Indeed, if $\Omega \subset \mathbb{R}^N$ is a domain of m revolution then one can show that the problem (2) admits a positive solution u of the form $u(x) = v(t_1,...,t_m)$ for some function $v : \tilde{\Omega} \subset \mathbb{R}^m \to \mathbb{R}$. Moreover, by imposing the extra condition that $\tilde{\Omega}$ is the unit cube in $\mathbb{R}^m$, we shall be able to look for solutions with certain properties that allows us to go well beyond the Sobolev critical exponent of $\mathbb{R}^N$. The next definition is the first step toward achieving this goal.*

Definition 1.3. *Suppose that Ω is a domain of m revolution in $\mathbb{R}^N$ with $\tilde{\Omega} = (0,1)^m := Q_m$ and $\sum_{i=1}^m n_i = N$. We denote by K_m, the set of all nonnegative functions $u \in H_G^1(\Omega)$ where $u = u(t_1,...,t_m)$ is increasing with respect to each component, i.e.,*

$$K_m(n_1,...,n_m) := \left\{ u \in H_G^1(\Omega) : u, u_{t_k} \geq 0 \ \text{a.e. in } Q_m \text{ for } 1 \leq k \leq m \ \right\}.$$

To shorten the notation, when there is no confusion, we just write K_m instead of $K_m(n_1,...,n_m)$.

Note that K_m is a closed convex cone in $H_G^1(\Omega)$.

Assumptions on Ω, f and a. We shall assume that Ω is a domain of m revolution in $\mathbb{R}^N$ which further satisfies

$$\tilde{\Omega} = (0,1)^m =: Q_m. \tag{3}$$

We now consider some assumptions on the nonlinearity f and $a(x)$.

A_1: $f \in C^1([0,\infty))$, $f(0) = f'(0) = 0$ and f is strictly increasing.

A_2: There exist $p > 2$ and $C > 0$ such that

$$|f(t)| \leq C(1 + |t|^{p-1}), \qquad \forall t \geq 0.$$

A_3: There exists $\mu > 2$ such that for all $t \in \mathbb{R}$ and $F(t) := \int_0^{|t|} f(s)\, ds$, we have

$$|t| f(|t|) \geq \mu F(t).$$

Also there exists $l > 1$ such that $2lF(t) \leq F(lt)$ for all $t \in \mathbb{R}$.

A_4: $a(x) > 0$ for a.e. $x \in \Omega$ and $a \in K_m \cap L^\infty(\Omega)$.

Finally, based on the number of revolutions m of the domain $\Omega \subset \mathbb{R}^N$, we define the m dimensional critical Sobolev exponent by $2_m^* := \frac{2m}{m-2}$ for $m \geq 3$ and $2_1^* = 2_2^* = \infty$. Note that $2_m^* := \frac{2m}{m-2}$ is greater than the standard Sobolev critical exponent for $\Omega \subset \mathbb{R}^N$. Indeed, if $N \geq 3$ then $2_N^* \leq 2_m^*$ as $m \leq N$.

We now state our existence theorems regarding (2).

Theorem 1.4. *Suppose that $\Omega \subset \mathbb{R}^N$ is a domain of m revolution with $\sum_{i=1}^m n_i = N$ and which satisfies (3). Suppose $A_1 - A_4$ hold with $p < 2_m^*$ in A_2. Then problem (2) admits at least one positive solution $u \in K_m$.*

An immediate corollary of this is the following where Ω has the desired symmetry and where a does not depend fully on all m variables.

Corollary 1.5. *Suppose Ω is a domain of m revolution which satisfies (3). Assume that*

$$a(t_1, t_2, ..., t_m) = a(t_1, t_2, ..., t_i)$$

for some $1 \leq i < m$. Suppose $A_1 - A_4$ hold with $p < 2_i^*$ in A_2. Then problem (2) admits at least one positive solution $u \in K_m$.

Example 1.6. *Consider the Neumann problem*

$$\begin{cases} -\Delta u + u = |x|^\alpha |u|^{p-2}u, & x \in B_1 \\ u > 0, & x \in B_1, \\ \frac{\partial u}{\partial \nu} = 0, & x \in \partial B_1, \end{cases} \tag{4}$$

where B_1 is the unit ball centered at the origin in $\mathbb{R}^N$, $N \geq 3$. Note that assumptions $A_1 - A_4$ in Theorem 1.4 hold for all $p > 2$ and $\alpha > 1$, and therefore problem (4) has a radially increasing solution $u(|x|)$.

Example 1.7. *Let $\Omega \subset \mathbb{R}^N$ with $N \geq 3$ be a domain of double revolution with $\tilde{\Omega} = (0,1)^2$, i.e.*

$$\Omega = \left\{(x_1, ..., x_N) \in \mathbb{R}^N;\ x_1^2 + ... + x_m^2 < 1 \text{ and } x_{m+1}^2 + ... + x_N^2 < 1\right\},$$

for some $1 \leq m < N$. Let $b_1, b_2 : [0,1] \to (0, \infty)$ be two functions that are twice differentiable and increasing. Consider the following problem

$$\begin{cases} -\Delta u + u = u^{p-1} b_1(\sqrt{x_1^2+\cdots+x_m^2}) b_2(\sqrt{x_{m+1}^2+\cdots+x_N^2}), & in\ \Omega, \\ u > 0, & in\ \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & on\ \partial\Omega, \end{cases} \tag{5}$$

It follows from Theorem 1.4, for each $p > 2$, problem (5) has a solution u of the form $u(x) = v(s,t)$ for some $v : [0,1] \times [0,1] \to \mathbb{R}^+$ with

$$s = \sqrt{x_1^2+\cdots+x_m^2}, \qquad t = \sqrt{x_{m+1}^2+\cdots+x_N^2}.$$

Moreover, the maps $s \to v(s,t)$ and $t \to v(s,t)$ are increasing.

1.2 Outline of approach

Our plan is to prove existence for (2) by making use of a new abstract variational principle established recently in [27] (see also [28, 29, 30]). To be more specific, let V be a reflexive Banach space, V^* its topological dual and K be a closed convex subset of V. Assume that $\Phi : V \to \mathbb{R}$ is convex, Gâteaux differentiable (with Gâteaux derivative $D\Phi(u)$) and lower semi-continuous and that $\Lambda : Dom(\Lambda) \subset V \to V^*$ is a linear symmetric operator. Let Φ^* be the Fenchel dual of Φ, i.e.

$$\Phi^*(u^*) = \sup\{\langle u^*, u\rangle - \Phi(u); u \in V\}, \qquad u^* \in V^*,$$

where the pairing between V and V^* is denoted by $\langle .,. \rangle$. Define the function $\Psi_K : V \to (-\infty, +\infty]$ by

$$\Psi_K(u) = \begin{cases} \Phi^*(\Lambda u), & u \in K, \\ +\infty, & u \notin K. \end{cases} \tag{6}$$

Consider the functional $I_K : V \to (-\infty, +\infty]$ defined by

$$I_K(w) := \Psi_K(w) - \Phi(w).$$

A point $u \in Dom(\Psi_K)$ is said to be a critical point of I_K if $D\Phi(u) \in \partial\Psi_K(u)$ or equivalently,

$$\Psi_K(v) - \Psi_K(u) \geq \langle D\Phi(u), v - u\rangle, \qquad \forall v \in V.$$

We shall now recall the following variational principle established in [27].

Theorem 1.8. *Let V be a reflexive Banach space and K be a closed convex subset of V. Let $\Phi : V \to \mathbb{R}$ be a Gâteaux differentiable convex and lower semi-continuous function, and let the linear operator $\Lambda : Dom(\Lambda) \subset V \to V^*$ be symmetric and positive. Assume that u is a critical point of $I_K(w) = \Psi_K(w) - \Phi(w)$, and that there exists $v \in K$ satisfying the linear equation,*

$$\Lambda v = D\Phi(u).$$

Then $u \in K$ is a solution of the equation

$$\Lambda u = D\Phi(u). \tag{7}$$

Before adapting this theorem to our case we make a couple of important observations. Firstly note that I_K (even if we pick $K = V$) is not the usual Euler-Lagrange energy associated with (7). The second point is that by picking K appropriately one can gain compactness; note the smaller we pick K the more manageable I_K becomes which makes proving the existence of critical points of I_K easier. But this needs to be balanced with the second part of the Theorem 1.8 where we need to solve the linear equation.

We now consider our case and for the purposes of clarity we consider the special case of $f(u) = |u|^{p-2}u$ and let us assume $p > 2$. Suppose that Ω is a domain of m revolution and then we write (2) in the abstract form

$$\Lambda u = D\Phi(u), \tag{8}$$

where Λ is the linear operator $-\Delta + 1$ and, Φ is a suitable Gâteaux differentiable convex and lower semi-continuous function. It can be easily seen that one should choose Φ to be

$$\Phi(w) := \frac{1}{p}\int_\Omega a(x)|w|^p dx.$$

One can then perform the calculations to see that I (we are omitting our choice of K for now) will be

$$I(w) = \frac{1}{q}\int_\Omega a(x)^{1-q}|-\Delta w + w|^q\,dx - \frac{1}{p}\int_\Omega a(x)|w|^p\,dx$$

where $q := p/(p-1)$ is the conjugate of p and where we are using the L^2 inner product as our V, V^* duality pairing (even though we have not specified V yet). Since the nonlinearity f is supercritical one is unable to find critical points of I on $H^1(\Omega)$ using standard variational approaches; for instance using a mountain-pass approach. To alleviate the problems introduced by the supercritical nonlinearity we work on the cone K_m. Using the monotonicity of the elements of K_m one obtains improved Sobolev imbeddings theorems (see Lemma 2.2) and this allows us to find a critical point u of I on K_m. To conclude that u is indeed a solution of (8), we then use Corollary 4.1 (an explicit version of Theorem 1.8).

1.3 Background when Ω is a ball in $\mathbb{R}^N$.

We now give a background of problems related to (2) in the case of supercritical nonlinearities. In all works that we mention Ω is given by B_1 (the unit ball in $\mathbb{R}^N$ centered at the origin). We mention that there are supercritical works related to (2) in the case of nonradial domains but they are generally problems which contain a small parameter ε.

In [1] they considered the variant of (2) given by $-\Delta u + u = |x|^\alpha u^p$ in B_1 with $\frac{\partial u}{\partial \nu} = 0$ on ∂B_1. They prove the existence of a positive radial solutions of this equation with arbitrary growth using a shooting argument. The solution turns out to be an increasing function. They also perform numerical computations to see the existence of positive oscillating solutions. In [34] they considered (2) along with the classical energy associated with the equation given by

$$E(u) := \int_{B_1} \frac{|\nabla u|^2 + u^2}{2}\,dx - \int_{B_1} a(|x|)F(u)\,dx,$$

where $F'(u) = f(u)$. Their goal was to find critical points of E over $H^1_{rad}(B_1) := \{u \in H^1(B_1) : u \text{ is radial}\}$. Of course since f is supercritical the standard approach of finding critical points will present difficulties and hence their idea was to find critical points of E over the cone $\{u \in H^1_{rad}(B_1) : 0 \leq u,\ u \text{ increasing}\}$. Doing this is somewhat standard but now the issue is the critical points don't necessarily correspond to critical points over $H^1_{rad}(B_1)$ and hence one can't conclude the critical points solve the equation. The majority of their work is to show that in fact the critical points of E on the cone are really critical points over the full space. In [20],

$$\begin{cases} -\Delta u + V(|x|)u = |u|^{p-2}u, & \text{in } B_1 \\ u > 0, & \text{in } B_1, \end{cases} \tag{9}$$

was examined under both homogeneous Neumann and Dirichlet boundary conditions. We will restrict our attention to their results regarding the Neumann boundary conditions. Consider $G(r,s)$ the Green function of the operator

$$\mathcal{L}(u) = -u'' - \frac{N-1}{r}u' + V(r)u, \quad u'(0) = 0,$$

with $u'(1) = 0$. Define now $H(r) := (G(r,r))^{-1}|\partial B_1|r^{N-1}$ for $0 < r \leq 1$. One of their results states that for $V \geq 0$ (not identically zero) if H has a local minimum at $\overline{r} \in (0,1]$ then for p large enough, (9) has a solution with Neumann boundary conditions and the solutions have a prescribed asymptotic behavior as $p \to \infty$. Additionally they can find as many solutions as H has local minimums. This work contains many results and we will list one of which more related. For $V = \lambda > 0$, the problem (9) has a positive nonconstant solution with Neumann boundary conditions provided p is large enough. This methods used in [20] appear to be very different from the methods used in the all the other works. It appears the works of [34] and [20] were done completely independent of each other. The next work related to (2) was [6] where they considered

$$\begin{cases} -\Delta u + b(|x|)x \cdot \nabla u + u = a(|x|)f(u), & \text{in } B_1 \\ u > 0, & \text{in } B_1, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial B_1, \end{cases} \tag{10}$$

where f is a supercritical nonlinearity and where various assumptions were imposed on b. Their approach was similar to [34] in the sense that they also worked on the cone $\{u \in H^1_{rad}(B_1) : 0 \leq u,\ u \text{ increasing}\}$ but instead of using a variational approach they used a topological approach. They were able to weaken the assumptions needed on f. In the case of $a = 1$ one sees that the constant u_0 is a solution provided $f(u_0) = u_0$. In [6] they have showed that (10) has a positive nonconstant solution in the case of $b = 0$ provided there is some $u_0 > 0$ with $f(u_0) = u_0$ and $f'(u_0) > \lambda_2^{rad}$ which is the second radial eigenvalue of $-\Delta + I$ in the unit ball with Neumann boundary conditions. Note that this result shows there is a positive nonconstant solution of (2) provided $p - 1 > \lambda_2^{rad}$. In [7] they considered various elliptic systems of the form

$$\begin{cases} -\Delta u + u = f(|x|, u, v), & \text{in } B_1 \\ -\Delta v + v = g(|x|, u, v), & \text{in } B_1 \\ \frac{\partial u}{\partial \nu} = \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial B_1. \end{cases}$$

In particular they examined the gradient system when $f(|x|,u,v) = G_u(|x|,u,v), g(|x|,u,v) = G_v(|x|,u,v)$ and they also considered the Hamiltonian system version where $f(|x|,u,v) = H_v(|x|,u,v), g(|x|,u,v) = H_u(|x|,u,v)$. In both cases there obtain positive solutions under various assumptions (which allowed supercritical nonlinearities). They also obtain positive nonconstant solutions in the case of $f(|x|,u,v) = f(u,v)$, $g(|x|,u,v) = g(u,v)$; note in this case there is the added difficulty of avoiding the possible constant solutions.

These results were extended to p-Laplace versions in [36]. The methods of [20] were extended to prove results regarding multi-layer radials solutions in [4]. We also mention the work of [8] where problems on the annulus were considered. We also mention the very recent works which extend some results and answer some open questions; see [2, 3, 5, 10, 25, 26].

In [12] we considered (2) in the case of $f(u) = |u|^{p-1}u$. Using a new variational principle we obtained positive solutions of (2); assuming the same assumptions as the earlier works. In the case of $a(x) = 1$ we obtain the existence of a positive nonconstant solution of (2). We remark our approach allowed us to deal directly with the supercritical nonlinearity without the need to cut the nonlinearity off.

We mention is that there is another type of supercritical problem that one can examine on B_1. One can examine supercritical equations like (2) or the case of zero Dirichlet boundary conditions when a is radial and $a = 0$ at the origin; a well known case of this is the Hénon equation given by $-\Delta u = |x|^\alpha u^p$ in B_1 with $u = 0$ on ∂B_1 where $0 < \alpha$. In [31] it was shown the Hénon equation has a positive solution if and only if $p < \frac{N+2+2\alpha}{N-2}$, and note this includes a range of supercritical p. This increased range of p is coming from the fact that $a = 0$ at the origin. We mention this phenomena is very different than what is going on in the above works. Results regarding positive solutions of supercritical Hénon equations on general domains have also been obtained, see [11] and [18].

One final point we mention is that there has been extensive study of subcritical, critical and supercritical Neumann problems on general domains in the case of (2) when $a = 1$ and where the equation involves a parameter that is sent to either zero or infinity. We have not attempted to discuss this problem but the interested reader should consult, for instance, [13, 33, 19, 21, 23, 22, 24, 38].

2 Elliptic problems on domains of m revolution

In this section we discuss the issue of solving equations on domains of m revolution in $\mathbb{R}^N$. We begin with the standard definition of a weak solution to a Neumann boundary value problem.

Definition 2.1. *We say v is a weak solution of*

$$\left\{ \begin{array}{rcll} -\Delta v + v &=& h(x) & \text{in } \Omega, \\ \partial_\nu v &=& 0 & \text{on } \partial\Omega, \end{array} \right. \tag{11}$$

provided $v \in H^1(\Omega)$ and satisfies

$$\int_\Omega \nabla v \cdot \nabla \eta + v\eta \, dx = \int_\Omega h(x)\eta, \qquad \forall \eta \in H^1(\Omega).$$

Given $\Omega \subset \mathbb{R}^N$ which is a domain of m revolution with $\sum_{i=1}^m n_i = N$ and, a function $h : \Omega \to \mathbb{R}$ that has symmetry compatible with Ω, i.e. $h(x)$ depends on just $t_1, ..., t_m$ (we write this as $h = h(t_1, ..., t_m)$), it is natural to look for a solution of (11) satisfying the same symmetry properties. Recall that

$$G = O(n_1) \times O(n_2) \times ... \times O(n_m),$$

and

$$H^1_G = \left\{ u \in H^1(\Omega) : gu = u \quad \forall g \in G \right\},$$

where $gu(x) := u(g^{-1}x)$. To find a solution of (11) it is sufficient (using the principle of symmetric criticality) to find a critical point of

$$E_\Omega(v) := \frac{1}{2} \int_\Omega |\nabla v|^2 + v^2 dx - \int_\Omega hv dx,$$

over $H^1_G(\Omega)$; i.e. to find a $v \in H^1_G(\Omega)$ such that

$$\int_\Omega \nabla v \cdot \nabla \eta + v\eta \, dx = \int_\Omega h(x)\eta \, dx, \ \ \forall \eta \in H^1_G(\Omega). \tag{12}$$

Note that we can identify $H^1_G(\Omega)$ with Y_m where $Y_m := \{v : \tilde{\Omega} \to \mathbb{R} : \|v\|_{Y_m} < \infty\}$ with

$$\|v\|^2_{Y_m} = \int_{\tilde{\Omega}} \left(\sum_{k=1}^m v^2_{t_k} + v^2 \right) d\mu_m(t), \qquad d\mu_m(t) = \prod_{k=1}^m t_k^{n_k - 1} dt_k.$$

Note we are using here that $v_{x_i} = v_{t_1} \frac{x_i}{t_1}$ for $1 \le i \le n_1$; $v_{x_i} = v_{t_2} \frac{x_i}{t_2}$ for $n_1 + 1 \le i \le n_2$ and we can carry on like this. So from this we see that $|\nabla_x v|^2 = \sum_{k=1}^m v^2_{t_k}$.
Note that Y_m is not $H^1(\tilde{\Omega})$ after noting the degenerate weights in $d\mu_m$. Also note that if $v \in H^1_G(\Omega)$ satisfies (12) then given $\eta \in H^1_G(\Omega)$ we have

$$\begin{aligned} c(n_1, ..., n_m) \int_{\tilde{\Omega}} h\eta \, d\mu_m(t) &= \int_\Omega h\eta dx \\ &= \int_\Omega \nabla v \cdot \nabla \eta + v\eta \, dx \\ &= c(n_1, ..., n_m) \int_{\tilde{\Omega}} \left(\sum_{k=1}^m v_{t_k}\eta_{t_k} + v\eta \right) d\mu_m(t), \end{aligned} \tag{13}$$

for all $\eta \in H^1_G(\Omega)$; note we are identifying $H^1_G(\Omega)$ and Y_m without changing notation. So we see that $v \in Y_m$ satisfies

$$\int_{\tilde{\Omega}} \left(\sum_{k=1}^m v_{t_k}\eta_{t_k} + v\eta \right) d\mu_m(t) = \int_{\tilde{\Omega}} h\eta \, d\mu_m(t), \quad \forall \eta \in Y_m. \tag{14}$$

Notation. For notational convenience we set

$$(\nabla_t v)_k = v_{t_k} \text{ for } 1 \le k \le m, \qquad \Delta_t v = \sum_{k=1}^m v_{t_k t_k}. \tag{15}$$

Integrating the (14) by parts formally one sees that v_m satisfies

$$0 = \int_{\tilde{\Omega}} \left(h + \sum_{k=1}^{m} \left\{ v_{t_k t_k} + \frac{n_k - 1}{t_k} v_{t_k} \right\} - v \right) \eta d\mu_m(t) \tag{16}$$

$$- \int_{\partial\tilde{\Omega}} \eta \left(\sum_{k=1}^{m} v_{t_k} \nu^k \right) \prod_{i=1}^{m} t_i^{n_i - 1}, \tag{17}$$

where $\nu = (\nu^1, ..., \nu^m)$ is the outward pointing normal on $\partial\tilde{\Omega}$. From this we see that v should satisfy

$$\begin{cases} -\Delta_t v - \sum_{k=1}^{m} \frac{n_k - 1}{t_k} v_{t_k} + v = h(t) & \text{in } \tilde{\Omega}, \\ \partial_\nu v = 0 & \text{on } \partial\tilde{\Omega}. \end{cases} \tag{18}$$

Note that the boundary condition on the 'inner boundaries' $t_k = 0$ for $1 \le k \le m$ is not coming from the weak formulation of the problem but rather from the symmetry of v and this requires v to have sufficient regularity.

We now prove a result regarding monotonicity of solutions of (11) provided h is monotonic. This result is crucial when applying Theorem 1.8, in particular when showing if $u \in K$ is a critical point of I_K over K then there is some $v \in K$ which satisfies $\Lambda v = D\Phi(u)$.

Proposition 2.1. *Assume Ω is domain of m revolution in $\mathbb{R}^N$ which satisfies (3) and suppose $0 \le h \in K_m \cap C^{1,1}(\overline{\Omega})$. Then there exists a unique $v \in K_m \cap C^{2,\alpha}(\overline{\Omega})$ (any $0 < \alpha < 1$) which satisfies (11).*

We shall provide a proof for Proposition 2.1 in the Appendix. We conclude this section by proving some improved imbeddings for functions in K_m given in Definition 1.3. Indeed, working on K_m, will allow us to obtain improved Sobolev critical exponents that are essential to consider supercritical problems from a variational point of view.

Lemma 2.2. *(Improved Sobolev imbeddings on K_m) Suppose Ω is a domain of m revolution in $\mathbb{R}^N$ which satisfies (3).*

1. *For $m \ge 3$ and for $1 \le q \le 2_m^*$ (in the case of $m = 1, 2$ for $1 \le q < \infty$) there is some C_q such that*
$$\|u\|_{L^q(Q_m)} \le C_q \|u\|_{H^1(\Omega)} \qquad \forall u \in K_m. \tag{19}$$
2. *For $m \ge 3$ and for $1 \le q \le 2_m^*$ (in the case of $m = 2$ for $1 \le q < \infty$) there is some C_q such that*
$$\|u\|_{L^q(\Omega)} \le C_q \|u\|_{H^1(\Omega)} \qquad \forall u \in K_m. \tag{20}$$

Proof. We first prove 1). The second part is a direct consequence of part 1).

1. Consider $A_1 := (2^{-1}, 1)^m$ and note we can decompose Q_m into the union 2^m disjoint cubes which are translations of A_1 (say $A_1, A_2, ..., A_{2^m}$) where we are missing a set of

measure zero of Q_m. Let $u \in K_m$ and let $1 \le q < \infty$ if $m = 2$ and for $m \ge 3$ let $1 \le q \le 2_m^*$. Then by the m dimensional Sobolev imbedding we have

$$\|u\|_{L^q(A_1)} \le C_q \|u\|_{H^1(A_1)}.$$

Note there is some constant $C = C(n_1, n_2, ..., n_m)$ such that $\|u\|_{H^1(A_1)} \le C\|u\|_{Y_m} = \tilde{C}\|u\|_{H^1(\Omega)}$. By monotonicity of u we have $\|u\|^q_{L^q(A_k)} \le \|u\|^q_{L^q(A_1)} \le \tilde{C}^q \|u\|^q_{H^1(\Omega)}$ for $1 \le k \le m$. Summing in k gives (19).

2. Note that for any $0 \le u$ with $u = u(t_1, ..., t_m)$ we have

$$\|u\|^q_{L^q(\Omega)} = C \int_{Q_m} u^q d\mu_m(t) \le C \int_{Q_m} u^q dt_1, ..., dt_m,$$

and we can then use the part 1 of this lemma to obtain the desired result.

□

3 Preliminaries from convex analysis

In this section we recall some important definitions and results from convex analysis and minimax principles for lower semi-continuous functions.

Let V be a real Banach space and V^* its topological dual and let $\langle ., . \rangle$ be the pairing between V and V^*. The weak topology on V induced by $\langle ., . \rangle$ is denoted by $\sigma(V, V^*)$. A function $\Psi : V \to \mathbb{R}$ is said to be weakly lower semi-continuous if

$$\Psi(u) \le \liminf_{n \to \infty} \Psi(u_n),$$

for each $u \in V$ and any sequence u_n approaching u in the weak topology $\sigma(V, V^*)$. Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be a proper convex function. The subdifferential $\partial\Psi$ of Ψ is defined to be the following set-valued operator: if $u \in Dom(\Psi) = \{v \in V;\ \Psi(v) < \infty\}$, set

$$\partial\Psi(u) = \{u^* \in V^*; \langle u^*, v - u\rangle + \Psi(u) \le \Psi(v) \text{ for all } v \in V\}$$

and if $u \notin Dom(\Psi)$, set $\partial\Psi(u) = \varnothing$. If Ψ is Gâteaux differentiable at u, denote by $D\Psi(u)$ the derivative of Ψ at u. In this case $\partial\Psi(u) = \{D\Psi(u)\}$.
The Fenchel dual of an arbitrary function Ψ is denoted by Ψ^*, that is function on V^* and is defined by

$$\Psi^*(u^*) = \sup\{\langle u^*, u\rangle - \Psi(u); u \in V\}.$$

Clearly $\Psi^* : V^* \to \mathbb{R} \cup \{\infty\}$ is convex and weakly lower semi-continuous. The following standard result is crucial in the subsequent analysis (see [15, 14] for a proof).

Proposition 3.1. *Let $\Psi : V \to \mathbb{R} \cup \{\infty\}$ be an arbitrary function. The following statements hold:*
(1) $\Psi^{**}(u) \le \Psi(u)$ for all $u \in V$.
(2) $\Psi(u) + \Psi^*(u^*) \ge \langle u^*, u\rangle$ for all $u \in V$ and $u^* \in V^*$.
(3) If Ψ is convex and lower-semi continuous then $\Psi^{**} = \Psi$ and the following assertions are equivalent:

- $\Psi(u) + \Psi^*(u^*) = \langle u, u^* \rangle$.
- $u^* \in \partial\Psi(u)$.
- $u \in \partial\Psi^*(u^*)$.

The above Proposition shows that for a convex lower semi-continuous function Ψ one has

$$\partial\Psi^* = \big(\partial\Psi\big)^{-1}.$$

We shall now recall some notations and results for the minimax principles of lower semi-continuous functions.

Definition 3.1. *Let V be a real Banach space, $\varphi \in C^1(V, \mathbb{R})$ and $\psi : V \to (-\infty, +\infty]$ be proper (i.e. $Dom(\psi) \neq \emptyset$), convex and lower semi-continuous. A point $u \in V$ is said to be a critical point of*

$$I := \psi - \varphi \tag{21}$$

if $u \in Dom(\psi)$ and if it satisfies the inequality

$$\langle D\varphi(u), u - v\rangle + \psi(v) - \psi(u) \geq 0, \qquad \forall v \in V. \tag{22}$$

Definition 3.2. *We say that the functional $I = \psi - \varphi$, given in (21), satisfies the Palais-Smale compactness condition (PS) if every sequence $\{u_n\}$ such that*

- $I[u_n] \to c \in \mathbb{R}$,
- $\langle D\varphi(u_n), u_n - v\rangle + \psi(v) - \psi(u_n) \geq -\epsilon_n \|v - u_n\|, \qquad \forall v \in V.$

where $\epsilon_n \to 0$, then $\{u_n\}$ possesses a convergent subsequence.

The following is proved in [37].

Theorem 3.3. *(Mountain Pass Theorem). Suppose that $I : V \to (-\infty, +\infty]$ is of the form (21) and satisfies the Palais-Smale condition and the Mountain Pass Geometry (MPG):*

1. *$I(0) = 0$.*
2. *there exists $e \in V$ such that $I(e) \leq 0$.*
3. *there exist $\alpha > 0$ and $0 < \rho < \|e\|$ such that for every $u \in V$ with $\|u\| = \rho$ one has $I(u) \geq \alpha$.*

Then I has a critical value $c \geq \alpha$ which is characterized by

$$c = \inf_{g \in \Gamma} \sup_{t \in [0,1]} I[g(t)],$$

where $\Gamma = \{g \in C([0,1], V) : g(0) = 0, g(1) = e\}$.

4 Existence results

In this section we assume that $\Omega \subset \mathbb{R}^N$ is a domain of m revolution with $\sum_{i=1}^m n_i = N$ and $\tilde{\Omega} = Q_m$. Throughout this section we always assume that assumptions $A_1 - A_4$ hold. For a function $f : [0, \infty) \to \mathbb{R}$ satisfying $A_1 - A_3$, define $F : \mathbb{R} \to \mathbb{R}$ and by

$$F(t) = \int_0^{|t|} f(s)\, ds,$$

and let $F^* : \mathbb{R} \to (-\infty, +\infty]$ be the Fenchel dual of F, i.e.,

$$F^*(s) = \sup_{t \in \mathbb{R}} \{ts - F(t)\}.$$

To adapt Theorem 1.8 in our case, consider the Banach space $V = H^1(\Omega) \cap L^p(\Omega)$, where $2 < p < 2^*_m$ and V is equipped with the following norm

$$\|v\| := \|v\|_{H^1(\Omega)} + \|v\|_{L^p(\Omega)}.$$

Assuming V^* is the topological dual of V, the pairing $\langle ., . \rangle$ between V and V^* is defined by

$$\langle v, v^* \rangle = \int_\Omega v(x) v^*(x)\, dx, \qquad \forall v \in V, \forall v^* \in V^*.$$

For $v \in V$ define the operator $A : Dom(A) \subset V \to V^*$ by $Av := -\Delta v + v$, where

$$Dom(A) = \{v \in V; \frac{\partial v}{\partial n} = 0, \quad \& \quad Av \in V^*\}.$$

Note that one can rewrite the problem (2) as

$$Au = D\varphi(u),$$

where $\varphi : L^p(\Omega) \to \mathbb{R}$ is defined by

$$\varphi(u) = \int_\Omega a(x) F(u) dx. \tag{23}$$

Denote by q the conjugate of p, i.e. $1/p + 1/q = 1$. Recall the set K_m, given in Definition 1.3, and define $\psi : V \to (-\infty, \infty]$, by

$$\psi(u) = \begin{cases} \int_\Omega a(x) F^*(\frac{-\Delta u + u}{a(x)})\, dx, & u \in K_m \cap W^{2,q}(\Omega) \\ +\infty, & u \notin K_m \cap W^{2,q}(\Omega), \end{cases} \tag{24}$$

with $Dom(\psi) = \{u \in V;\, \psi(u) < \infty\}$. In Lemma 4.4, we shall show that ψ is convex and weakly lower semi-continuous.

The following result is a direct consequence of Theorem 1.8. However, for the convenience of the reader, we shall also prove it in this paper.

Corollary 4.1. *Assume that u is a critical point of*

$$I(w) := \psi(w) - \varphi(w), \tag{25}$$

where ψ and φ are given in (24) and (23) respectively. If there exists $v \in Dom(\psi)$ satisfying the linear equation,

$$\begin{cases} -\Delta v + v = a(x)f(u), & \text{in } \Omega, \\ \frac{\partial v}{\partial \nu} = 0, & \text{on } \partial\Omega, \end{cases} \tag{26}$$

then u is a solution of the equation

$$\begin{cases} -\Delta u + u = a(x)f(u), & \text{in } \Omega, \\ \frac{\partial u}{\partial \nu} = 0, & \text{on } \partial\Omega. \end{cases}$$

Proof. Since u is a critical point of I, it follows from Definition 3.1 that

$$\psi(w) - \psi(u) \geq \langle D\varphi(u), w - u\rangle, \qquad \forall w \in V. \tag{27}$$

Since $I(u)$ is finite we have that $u \in Dom(\psi)$ and

$$\psi(u) = \int_\Omega a(x)F^*\big(\frac{-\Delta u + u}{a(x)}\big)\, dx < \infty.$$

It then follows that $Au = -\Delta u + u \in L^q(\Omega)$ and $\psi(u) = \varphi^*(Au)$ as shown in Lemma 4.3. By assumption, there exists $v \in Dom(\psi)$ satisfying $Av = D\varphi(u)$. Substituting $w = v$ in (27) yields that

$$\varphi^*(Av) - \varphi^*(Au) = \psi(v) - \psi(u) \geq \langle D\varphi(u), v - u\rangle = \langle Av, v - u\rangle. \tag{28}$$

On the other hand it follows from $Av = D\varphi(u)$ and Proposition 3.1 that $u \in \partial\varphi^*(Av)$ from which we obtain

$$\varphi^*(Au) - \varphi^*(Av) \geq \langle u, Au - Av\rangle. \tag{29}$$

Adding up (28) with (29) we obtain

$$\langle u, Au - Av\rangle + \langle Av, v - u\rangle \leq 0.$$

Since A is symmetric we obtain that $\langle u - v, Au - Av\rangle \leq 0$ from which we obtain

$$\int_\Omega |\nabla u - \nabla v|^2\, dx + \int_\Omega |u - v|^2\, dx \leq 0,$$

thereby giving that $u = v$. It then follows that $Au = Av = D\varphi(u)$ as claimed. □

Evidently, Corollary 4.1 maps out the plan for the prove of Theorem 1.4. Indeed, by using Theorem 3.3, we show that the functional I defined in (25) has a nontrivial critical point and then we shall prove that the linear equation (26) has a solution.

We shall need some preliminary results before proving the main Theorems in Introduction. We first list some properties of the function F.

Lemma 4.2. *The following assertions hold:*

1. $F \in C^2(\mathbb{R})$.
2. $F : \mathbb{R} \to \mathbb{R}$ *is strictly convex.*
3. *There exists a constant* $C > 0$ *such that* $0 \leq F(t) \leq C(1 + |t|^p)$ *and* $\mu F(t) \leq tF'(t)$ *for all* $t \in \mathbb{R}$.
4. *There exists a constant* $C > 0$ *such that* $F(t) \geq Ct^\mu$ *and* $F'(t)t \geq Ct^\mu$ *for* $|t| \geq 1$.
5. $F^* \in C^1(\mathbb{R})$ *and*
$$\frac{\mu}{\mu - 1} F^*(s) \geq sF^{*\prime}(s), \qquad \forall s \in \mathbb{R}.$$
6. $F^*(s) \geq 0$ *for all* $s \in \mathbb{R}$ *and* $F^*(0) = 0$.
7. *There exists a constant* $L > 0$ *such that* $F^*(2s) \leq LF^*(s)$ *for all* $s \in \mathbb{R}$.

Proof. Part 1) simply follows from A_1. Part 2) is an immediate consequence of the fact $F^{''}(t) = f'(|t|) > 0$ for all $t \neq 0$ and $F^{''}(0) = f'(0) = 0$. Part 3) follows from A_2 and A_3. Part 4) follows from part 3) and A_1. We now proof part 5). The fact that $F^* \in C^1(\mathbb{R})$ follows from Theorem 26.6 in [35]. Take $s \in \mathbb{R}$ and let t_0 be a point that maximizes $sup_{t \in \mathbb{R}}\{ts - F(t)\}$. Thus $s = F'(t_0)$. Since F' is strictly monotone and hence invertible we have that $t_0 = (F')^{-1}(s) = F^{*\prime}(s)$. It now follows from $\mu F(t_0) \leq t_0 F'(t_0)$ and $F^*(s) = t_0 s - F(t_0)$ that

$$sF^{*\prime}(s) = st_0 = t_0 F'(t_0) \geq \mu F(t_0) = \mu t_0 s - \mu F^*(s) = \mu s F^{*\prime}(s) - \mu F^*(s)$$

from which we obtain

$$\mu F^*(s) \geq (\mu - 1) s F^{*\prime}(s),$$

as desired.
Part 6): Since

$$-F(0) = \inf_{s \in \mathbb{R}} F^*(s),$$

we obtain that $F^*(s) \geq 0$ for all $s \in \mathbb{R}$. Also as $0 = f(0) = F'(0) = \partial F(0)$, it follows from Proposition 3.1 that

$$F(0) + F^*(0) = 0.$$

Thus, $F^*(0) = 0$ as by the definition $F(0) = 0$.
Part 7): By A_3, there exists a constant $l > 0$ such that $2lF(t) \leq F(lt)$ for all $t \in \mathbb{R}$. In the context of Orlicz spaces this property is known as ∇_2 condition (see [32] for more details). It now follows from ∇_2 condition that for each $s \in \mathbb{R}$,

$$\begin{aligned} F^*(s) = \sup_{t \in \mathbb{R}} \{st - F(t)\} \geq \sup_{t \in \mathbb{R}} \left\{st - \frac{F(lt)}{2l}\right\} &= \frac{1}{2l} \sup_{t \in \mathbb{R}} \{2slt - F(lt)\} \\ &= \frac{1}{2l} \sup_{t \in \mathbb{R}} \{2st - F(t)\} = \frac{1}{2l} F^*(2s). \end{aligned}$$

□

Recall that q is the conjugate of p, i.e. $\frac{1}{q} + \frac{1}{p} = 1$.

Lemma 4.3. *Assume that $\varphi : V \to \mathbb{R}$ is defined by $\varphi(v) = \int_\Omega a(x)F(v)\,dx$. Let $\varphi^* : V^* \to (0, +\infty]$ be the Fenchel dual of φ. The following assertions hold.*

1. *For each $h \in L^q(\Omega)$ we have*
$$\varphi^*(h) = \int a(x)F^*\big(\frac{h(x)}{a(x)}\big)\,dx.$$

2. *There exist positive constants C_1 and C_2 such that*
$$\varphi^*(h) \geq C_1\|h\|^q_{L^q(\Omega)} - C_2$$
for all $h \in L^q(\Omega)$.

3. *The function φ is differentiable and $\langle D\varphi(u), u\rangle \geq \mu\varphi(u)$ for all $u \in V$.*

4. *Let $h \in Dom(\varphi^*)$. Then $(1+t)h \in Dom(\varphi^*)$ for all $0 \leq t \leq 1$. Moreover, the directional derivative*
$$D_h\varphi^*(h) := \lim_{t\to 0^+}\frac{\varphi^*(h+th) - \varphi^*(h)}{t},$$
exists and
$$0 \leq D_h\varphi^*(h) = \int_\Omega F^*\big(\frac{h}{a}\big)h\,dx \leq \frac{\mu}{\mu-1}\varphi^*(h).$$

Proof. *1.* Take $h \in L^q(\Omega)$. It follows from the density of V in $L^p(\Omega)$ that

$$\begin{aligned}
\varphi^*(h) &= \sup_{v\in V}\{\langle v, h\rangle - \varphi(v)\}\\
&= \sup_{v\in V}\Big\{\int_\Omega v(x)h(x)\,dx - \int_\Omega a(|x|)F(v)\Big\}\\
&= \sup_{v\in L^p(\Omega)}\Big\{\int_\Omega v(x)h(x)\,dx - \int_\Omega a(|x|)F(v)\Big\} = \int_\Omega a(x)F^*\big(\frac{h(x)}{a(x)}\big)\,dx,
\end{aligned}$$

where for the last equality we have used Proposition 2.1 in ([15], page 271) and the fact that $a(x) > 0$ for a.e. $x \in \Omega$.
2. It follows from the boundedness of the function a and part 3) of Lemma 4.2 that

$$\varphi(v) = \int_\Omega a(x)F(v)\,dx \leq C\int(1+|v|^p)\,dx$$

for some constant $C > 0$ and all $v \in L^p(\Omega)$. It then follows that

$$\begin{aligned}
\varphi^*(h) &= \sup_{v\in V}\{\langle v, h\rangle - \varphi(v)\}\\
&\geq \sup_{v\in V}\Big\{\int_\Omega v(x)h(x)\,dx - C\int_\Omega(1+|v|^p)\,dx\Big\}\\
&= \sup_{v\in L^p(\Omega)}\Big\{\int_\Omega v(x)h(x)\,dx - C\int_\Omega(1+|v|^p)\,dx\Big\}\\
&= C_1\|h\|^q_{L^q(\Omega)} - C_2
\end{aligned}$$

for some constants C_1 and C_2.
3. Differentiability of φ simply follows from A_2 and the fact that $a \in L^\infty(\Omega)$. An easy computation also shows that $D\varphi(u) = a(x)F'(u)$. It now follows from part 3) of Lemma 4.2 that

$$\langle D\varphi(u), u\rangle = \int_\Omega a(x)F'(u)u\,dx \geq \int_\Omega \mu a(x)F(u)\,dx = \mu\varphi(u).$$

4. It follows from part 7) of Lemma 4.2 that there exists a constant $L > 0$ such that F^* satisfies the following condition,

$$F^*(2s) \leq LF^*(s), \qquad \forall s \in \mathbb{R}. \tag{30}$$

Therefore, if $h \in Dom(\varphi^*)$ then $\varphi^*(2h) \leq L\varphi^*(h) < \infty$. On the other hand for each $0 < t < 1$ by the convexity of φ^* we have that

$$\varphi^*((1+t)h) = \varphi^*\big((1-t)h + t2h\big) \leq (1-t)\varphi^*(h) + t\varphi^*(2h) < \infty,$$

from which we have that $(1+t)h \in Dom(\varphi^*)$.

It follows from part 6) of Lemma 4.2 that $F^*(0) = \inf_{s\in\mathbb{R}} F^*(s)$. Since F^* is differentiable we must have $F^{*\prime}(0) = 0$. For all $s \in \mathbb{R}$ it follows from the monotonicity of $F^{*\prime}$ that

$$(F^{*\prime}(s) - F^{*\prime}(0))(s-0) \geq 0.$$

Thus,

$$sF^{*\prime}(s) \geq 0. \tag{31}$$

It now follows from the latter inequality and part 5) of Lemma 4.2 that for all $x \in \Omega$ with $a(x) \neq 0$ and $h(x) \in \mathbb{R}$, we have

$$0 \leq 2h(x)F^{*\prime}\big(\frac{2h(x)}{a(x)}\big) \leq \frac{\mu}{\mu-1}a(x)F^*\big(\frac{2h(x)}{a(x)}\big).$$

Integrating both sides yields that

$$0 \leq \int_\Omega 2h(x)F^{*\prime}\big(\frac{2h(x)}{a(x)}\big)\,dx \leq \frac{\mu}{\mu-1}\int_\Omega a(x)F^*\big(\frac{2h(x)}{a(x)}\big)\,dx = \frac{\mu}{\mu-1}\varphi^*(2h) < \infty.$$

It now follows from the monotonicity of $F^{*\prime}$ and the latter inequality that

$$0 \leq \int_\Omega h(x)F^{*\prime}\big(\frac{(1+t)h(x)}{a(x)}\big)\,dx \leq \int_\Omega h(x)F^{*\prime}\big(\frac{2h(x)}{a(x)}\big)\,dx \leq \frac{\mu}{2(\mu-1)}\varphi^*(2h) \tag{32}$$

By the convexity of F^* we have that the map

$$t \to \frac{aF^*\big(\frac{h+th}{a}\big) - aF^*\big(\frac{h}{a}\big)}{t}$$

is increasing on $(0,1)$ and from (32) we obtain that

$$0 \leq h(x)F^{*\prime}\big(\frac{h(x)}{a(x)}\big) \leq \frac{aF^*\big(\frac{h+th}{a}\big) - aF^*\big(\frac{h}{a}\big)}{t} \leq h(x)F^{*\prime}\big(\frac{(1+t)h(x)}{a(x)}\big) \leq h(x)F^{*\prime}\big(\frac{2h(x)}{a(x)}\big).$$

It now follows from the dominated convergence theorem and (32) that

$$\begin{aligned}\lim_{t\to 0^+}\frac{\varphi^*(h+th)-\varphi^*(h)}{t} &= \int_\Omega \lim_{t\to 0^+}\frac{aF^*\big(\frac{h+th}{a}\big)-aF^*\big(\frac{h}{a}\big)}{t}\,dx\\ &= \int_\Omega aF^{*\prime}(\frac{h}{a})\frac{h}{a}\,dx\\ &\le \int_\Omega \frac{\mu}{\mu-1}aF^*(\frac{h}{a})\,dx=\frac{\mu}{\mu-1}\varphi^*(h),\end{aligned}$$

from which the desired result follows. □

Lemma 4.4. *The functional* $\psi : V\to(-\infty,\infty]$ *defined by*

$$\psi(u)=\begin{cases}\int_\Omega a(x)F^*(\frac{-\Delta u+u}{a(x)})\,dx, & u\in K_m\cap W^{2,q}(\Omega)\\ +\infty, & u\notin K_m\cap W^{2,q}(\Omega),\end{cases}$$

is convex and weakly lower semi-continuous.

Proof. We first show that ψ is weakly lower semi-continuous. Let $\{u_n\}$ be a sequence in V that converges weakly to some $u\in V$. If $\alpha:=\liminf_{n\to\infty}\psi(u_n)=\infty$ the there is nothing to prove. Let us assume that $\alpha<\infty$. Thus, up to a subsequence, $u_n\to u$ a.e., $\psi(u_n)<\infty$ and $\lim_{n\to\infty}\psi(u_n)=\alpha$. Since $u_n\to u$ a.e. we have that $u\in K_m$. It follows from part 1) of Lemma 4.3 that $\psi(u_n)=\varphi^*(-\Delta u_n+u_n)$. It also follows from part 2) of Lemma 4.3 that $\{u_n\}$ is also bounded in $W^{2,q}(\Omega)$. Thus, up to a subsequence, we must have that $u_n\to u$ weakly in $W^{2,q}(\Omega)$ and therefore $u\in K_m\cap W^{2,q}(\Omega)$. Take $v\in L^p(\Omega)$. It follows that

$$\psi(u_n)=\varphi^*(-\Delta u_n+u_n)\ge\int_\Omega v(x)(-\Delta u_n+u_n)\,dx-\varphi(v),$$

from which we obtain

$$\liminf_{n\to\infty}\psi(u_n)=\liminf_{n\to\infty}\varphi^*(-\Delta u_n+u_n)\ge\int_\Omega v(x)(-\Delta u+u)\,dx-\varphi(v),$$

Taking sup over all $v\in L^p(\Omega)$ implies that

$$\liminf_{n\to\infty}\psi(u_n)=\liminf_{n\to\infty}\varphi^*(-\Delta u_n+u_n)\ge\varphi^*(-\Delta u+u)=\psi(u),$$

from which the lower semi-continuity of ψ follows.

We now show that ψ is convex. Let $u_1,u_2\in V$ and $t\in(0,1)$. We need to verify that

$$\psi\big(tu_1+(1-t)u_2\big)\le t\psi(u_1)+(1-t)\psi(u_2). \tag{33}$$

Note first that F^* is non-negative by Lemma 4.2, and $a\ge 0$ by assumption. Thus, we have that $\psi\ge 0$. If one of $\psi(u_1)$ or $\psi(u_2)$ is $+\infty$ then we are done. So assume that $\psi(u_1),\psi(u_2)\in\mathbb{R}$. It then follows that $u_1,u_2\in K_m\cap W^{2,q}(\Omega)$ and

$$\int_\Omega a(x)F^*\Big(\frac{-\Delta u_i+u_i}{a(x)}\Big)\,dx<\infty,\qquad i=1,2. \tag{34}$$

Since $K_m \cap W^{2,q}(\Omega)$ is a convex set we have that $tu_1 + (1-t)u_2 \in K_m \cap W^{2,q}(\Omega)$. On the other hand, for almost every $x \in \Omega$, it follows from the convexity of F^* and linearity of the map $u \to -\Delta u + u$ that

$$\begin{aligned} F^*\Big(\frac{-\Delta(tu_1 + (1-t)u_2) + tu_1 + (1-t)u_2}{a(x)}\Big) &= F^*\Big(\frac{t(-\Delta u_1 + u_1) + (1-t)(-\Delta u_2 + u_2)}{a(x)}\Big) \\ &\leq tF^*\Big(\frac{-\Delta u_1 + u_1}{a(x)}\Big) + (1-t)F^*\Big(\frac{-\Delta u_2 + u_2}{a(x)}\Big). \end{aligned}$$

Therefore, by multiplying the latter expression by $a(x)$ and integrating over Ω the inequity (33) follows. □

Lemma 4.5. *Let $\{u_n\} \subset K_m$ be a sequence in $H^1(\Omega)$ that converges weakly to some $u \in K_m$ and also $u_n \to u$ a.e.. Then*

1. *For each $2 \leq r < 2_m^*$ we have that $u_n \to u$ strongly in $L^r(\Omega)$.*
2. *$\varphi(u_n) \to \varphi(u)$.*
3. *$\langle D\varphi(u_n), u_n - u\rangle \to 0$.*
4. *For each $\delta > 0$, we have $\varphi(\delta u_n - \delta u) \to 0$.*

Proof. 1) Suppose $2 \leq r < 2_m^*$ and set $T := \min\{2_m^*, r+1\}$ (recall if $m = 2$ then $2_m^* = \infty$). By interpolating L^2 and L^T we have

$$\|u_n - u\|_{L^r} \leq \|u_n - u\|_{L^2}^{\theta} \|u_n - u\|_{L^T}^{1-\theta},$$

for some $0 < \theta \leq 1$. Since $u_n, u \in K$, it follows from Lemma 2.2 that $\|u_n - u\|_{L^T}$ is bounded. It now follows from the weak convergence in H^1 that $u_n \to u$ strongly in L^2, from which together with latter inequality we get that $u_n \to u$ strongly in $L^r(\Omega)$.

2) It follows from A_2 that $F(u_n) \leq C(1 + |u_n|^p)$. Thus, the result follows from part 1) and the dominated convergence theorem.

3) Note that

$$|\langle D\varphi(u_n), u_n - u\rangle| \leq \int a(|x|)|f(u_n)(u_n - u)|\, dx \leq C \int |u_n - u|(1 + |u_n|^{p-1})\, dx,$$

and by Holder inequality and the result of part 1) we obtain

$$\int |u_n - u|(1 + |u_n|^{p-1})\, dx \leq \int |u_n - u|(1 + |u_n|)^{p-1}\, dx \leq \|u_n - u\|_{L^p} \|1 + |u_n|\|_{L^p}^{\frac{p}{q}} \to 0.$$

4) It follows from A_2 that $F(\delta u_n - \delta u) \leq C(1 + |\delta u_n - \delta u|^p)$, which together with the dominated convergence theorem and part 1) the desired result follows. □

Proposition 4.1. *Consider the functional $I : V \to \mathbb{R} \cup \{+\infty\}$ defined by*

$$I(u) := \psi(u) - \varphi(u),$$

where $\varphi(u) = \int_\Omega a(x)F(u)\,dx$ and

$$\psi(u) = \begin{cases} \int_\Omega a(x)F^*(\frac{-\Delta u+u}{a(x)})\,dx, & u \in K_m \cap W^{2,q}(\Omega) \\ +\infty, & u \notin K_m \cap W^{2,q}(\Omega). \end{cases}$$

Then I has a nontrivial critical point.

Proof. We make use Theorem 3.3 to prove this lemma. First note that, by A_2, the functional φ is C^1 and

$$D\varphi(u) = a(x)f(u).$$

Note also that ψ is proper and convex as $K_m \cap W^{2,q}(\Omega)$ is convex in V. It also follows from Lemma 4.4 that ψ is weakly lower semi-continuous. We shall now proceed in several steps.

Step 1. In this step we shall verify the mountain pass geometry for I.
By Lemma 4.2 we have that $\mu F(t) \le tF'(t)$ and $\frac{\mu}{\mu-1}F^*(t) \ge tF^{*\prime}(t)$. Thus there exist constants C_1 and C_2 such that for $t \ge 1$,

$$F(t) \ge C_1|t|^\mu, \quad F^*(t) \le C_2|t|^{\frac{\mu}{\mu-1}}. \tag{35}$$

It is clear that $I(0) = 0$ as $F^*(0)$ by part 6) of Lemma 4.2. Since $e = \sup_{x\in\Omega} a(x)+1 \in K_m$, it follows that for $t \ge 1$

$$\begin{aligned} I(te) &= \int a(x)F^*(\frac{te}{a(x)})dx - \int_{B_1} a(|x|)F(te)dx \\ &\le C_2 \int a(x)^{1-\frac{\mu}{\mu-1}}|te|^{\frac{\mu}{\mu-1}}dx - C_1 \int a(|x|)|te|^\mu dx \end{aligned}$$

Now, since $\mu > 2$ one has that $\frac{\mu}{\mu-1} < 2$. Thus for t sufficiently large $I(te)$ is negative. We now prove condition 3) of (MPG). Take $u \in Dom(\psi)$ with $\|u\|_V = \rho > 0$. We have

$$I(u) = \psi(u) - \varphi(u) = \varphi^*(Au) - \varphi(u) \ge < Au, u > -2\varphi(u) = \|u\|^2_{H^1} - 2\varphi(u) \tag{36}$$

Since the function $a(x)$ is bounded it follows from A_2 that

$$\forall \delta > 0, \exists C_\delta > 0, \qquad |a(x)F(t)| \le \delta|t|^2 + C_\delta|t|^p, \tag{37}$$

from which we obtain

$$\forall \delta > 0, \exists C_\delta > 0, \qquad \varphi(u) = \int a(x)F(u) \le \delta\|u\|^2_{L^2(\Omega)} + C_\delta\|u\|^p_{L^p(\Omega)}. \tag{38}$$

Note that from Lemma 2.2, for $u \in K_m$ one has $\|u\|_{L^p} \le C_p\|u\|_{H^1}$ and therefore,

$$\|u\|_V = \|u\|_{H^1} + \|u\|_{L^p} \le (1 + C_p)\|u\|_{H^1} \tag{39}$$

for some constant $C > 0$. It now follows from (38) and (39) that

$$\begin{aligned} I(u) &= \|u\|_{H^1}^2 - 2\varphi(u) \geq \|u\|_{H^1}^2 - 2\delta\|u\|_{L^2(\Omega)}^2 - 2C_\delta\|u\|_{L^p(\Omega)}^p \\ &\geq (1-2\delta)\|u\|_{H^1}^2 - 2C_\delta C_p^p\|u\|_{H^1(\Omega)}^p \\ &\geq \frac{(1-2\delta)}{(1+C_p)^2}\|u\|_V^2 - 2C_\delta C_p^p\|u\|_V^p \\ &= \frac{(1-2\delta)}{(1+C_p)^2}\rho^2 - 2C_\delta C_p^p\rho^p \end{aligned}$$

Therefore, for $\delta < 1/2$ we have that

$$I[u] \geq \frac{(1-2\delta)}{(1+C_p)^2}\rho^2 - 2C_\delta C_p^p\rho^p > 0$$

provided $\rho > 0$ is small enough. If $u \notin Dom(\psi)$, then clearly $I(u) > 0$. Therefore (MPG) holds for the functional I.

Step 2. We verify Palais-Smale compactness condition. Suppose that $\{u_n\}$ is a sequence in K_m such that $I(u_n) \to c \in \mathbb{R}$ as $\epsilon_n \to 0$ and

$$\langle D\varphi(u_n), u_n - v\rangle + \psi(Av) - \psi(Au_n) \geq -\epsilon_n\|v - u_n\|_V, \qquad \forall v \in V. \tag{40}$$

We must show that $\{u_n\}$ has a convergent subsequence in V. First, note that $u_n \in Dom(\psi)$ and therefore,

$$I(u_n) = \varphi^*(Au_n) - \varphi(u_n) \to c, \qquad \text{as } n \to \infty.$$

Thus, for large values of n we have

$$\varphi^*(Au_n) - \varphi(u_n) \leq 1 + c. \tag{41}$$

Since $Au_n \in Dom(\varphi^*)$, it follows from part 4) of Lemma 4.3 that $(1+r)Au_n \in Dom(\varphi^*)$ for $0 \leq r \leq 1$. By setting $v = (1+r)u_n \in K_m \cap W^{2,q}(\Omega)$ for $0 < r \leq 1$ in (40) we have that

$$-\langle D\varphi(u_n), ru_n\rangle + \varphi^*(Au_n + rAu_n) - \varphi^*(Au_n) \geq -r\epsilon_n\|u_n\|_V. \tag{42}$$

Dividing both sides by r and letting $r \to 0^+$ yield that,

$$-\langle D\varphi(u_n), u_n\rangle + D_{Au_n}\varphi^*(Au_n) \geq -\epsilon_n\|u_n\|_V, \tag{43}$$

where $D_{Au_n}\varphi^*(Au_n)$ is the directional derivative of φ^* at Au_n in the direction Au_n that exists due to Lemma 4.3 part 4), and furthermore $D_{Au_n}\varphi^*(Au_n) \leq \frac{\mu}{\mu-1}\varphi^*(Au_n)$. Multiply (43) by $-1/2$ and sum it up with (41) to get

$$\varphi^*(Au_n) - \frac{1}{2}D_{Au_n}\varphi^*(Au_n) + \frac{1}{2}\langle D\varphi(u_n), u_n\rangle - \varphi(u_n) \leq 1 + c + \|u_n\|_V,$$

and therefore by 3) and 4) in Lemma 4.3 we obtain that

$$\big(1 - \frac{\mu}{2(\mu-1)}\big)\varphi^*(Au_n) + \big(\frac{\mu}{2} - 1\big)\varphi(u_n) \leq 1 + c + \|u_n\|_V. \tag{44}$$

Since $\mu > 2$ we have that

$$\frac{\mu}{2} - 1 > 0 \qquad \& \qquad 1 - \frac{\mu}{2(\mu-1)} > 0.$$

Taking now into account that $\varphi^*(Au_n) \geq 0, \varphi(u_n) \geq 0$, it follows from (44) that

$$\varphi^*(Au_n) + \varphi(u_n) \leq C_2(1 + \|u_n\|_V), \tag{45}$$

for an appropriate constant $C_2 > 0$. On the other hand

$$\varphi^*(Au_n) + \varphi(u_n) \geq \langle Au_n, u_n \rangle = \|u_n\|_{H^1}^2,$$

which according to (44) results in

$$\|u_n\|_{H^1}^2 \leq C_2(1 + \|u_n\|_V).$$

It also follows (39) that $\|u_n\|_V \leq (1 + C_p)\|u_n\|_{H^1}$ and therefore

$$\|u_n\|_{H^1}^2 \leq C_0(1 + \|u_n\|_{H^1}),$$

for some constant C_0. Therefore $\{u_n\}$ is bounded in $H^1(\Omega)$. By passing to a subsequence if necessary, there exists $\bar{u} \in H^1(\Omega)$ such that $u_n \rightharpoonup \bar{u}$ weakly in $H^1(\Omega)$, $u_n \to \bar{u}$ strongly in $L^2(\Omega)$ and $u_n \to \bar{u}$ a.e.. Note first that $\bar{u} \in K_m$. It also follows from (44) that $\{\varphi^*(Au_n)\}$ is bounded and therefore, by Lemma 4.4, we obtain

$$\varphi^*(A\bar{u}) \leq \liminf_{n\to\infty} \varphi^*(Au_n) < \infty,$$

from which one has $\bar{u} \in Dom(\psi)$. By setting $v = \bar{u}$ in (40) we obtain

$$-\langle D\varphi(u_n), \bar{u} - u_n \rangle + \varphi^*(A\bar{u}) - \varphi^*(Au_n) \geq -\epsilon_n \|\bar{u} - u_n\|_V, \tag{46}$$

By Lemma 4.5 we have that $\langle D\varphi(u_n), \bar{u} - u_n \rangle \to 0$, and by (39) we have that $\|u_n - \bar{u}\|_V$ is bounded. Therefore passing into limits in (46) results in

$$\limsup_{n\to\infty} \varphi^*(Au_n) \leq \varphi^*(A\bar{u}). \tag{47}$$

The latter inequality together with the fact that $\varphi^*(A\bar{u}) \leq \liminf_{n\to\infty} \varphi^*(Au_n)$ yield that

$$\varphi^*(A\bar{u}) = \lim_{n\to\infty} \varphi^*(Au_n).$$

Now observe that

$$\|u_n\|_{H^1}^2 - \|\bar{u}\|_{H^1}^2 = \langle Au_n, u_n \rangle - \langle A\bar{u}, \bar{u} \rangle = \langle Au_n, u_n - \bar{u} \rangle + \langle Au_n - A\bar{u}, \bar{u} \rangle. \tag{48}$$

But weakly convergence of u_n to $\bar{u}$ in $H^1(\Omega)$ means that $Au_n \rightharpoonup A\bar{u}$ weakly in $H^{-1}(\Omega)$, thus

$$\langle Au_n - A\bar{u}, \bar{u} \rangle \to 0, \qquad \text{as } n \to \infty. \tag{49}$$

Let $0 < \delta < 1$. It follows from Lemma 4.3 and part 6) of Lemma 4.2 that $\varphi^*(0) = 0$. Thus, by the convexity of φ^* we have that

$$\varphi^*(\delta A u_n) = \varphi^*\big(\delta A u_n + (1-\delta)0\big) \le \delta\varphi^*(Au_n) + (1-\delta)\varphi^*(0) = \delta\varphi^*(Au_n).$$

We then have that

$$\begin{aligned} |\langle Au_n, u_n - \bar{u}\rangle| &\le \varphi(\frac{u_n - \bar{u}}{\delta}) + \varphi^*(\delta A u_n) \\ &\le \varphi(\frac{u_n - \bar{u}}{\delta}) + \delta\varphi^*(Au_n). \end{aligned}$$

By taking lim sup as $n \to \infty$ we have that

$$\limsup_{n\to\infty} |\langle Au_n, u_n - \bar{u}\rangle| \le \limsup_{n\to\infty} \varphi(\frac{u_n - \bar{u}}{\delta}) + \delta \limsup_{n\to\infty} \varphi^*(Au_n)$$

Now by virtue of Lemma 4.5 we have that $\limsup_{n\to\infty} \varphi(\frac{u_n-\bar{u}}{\delta}) = 0$ from which we obtain

$$\limsup_{n\to\infty} |\langle Au_n, u_n - \bar{u}\rangle \le \delta\varphi^*(A\bar{u}).$$

By now letting $\delta \to 0$ we obtain that

$$\limsup_{n\to\infty} |\langle Au_n, u_n - \bar{u}\rangle| = 0. \tag{50}$$

Therefore, from (48), (49) and (50) one has

$$u_n \to \bar{u} \quad \text{strongly in} \quad H^1.$$

It now follows from Lemma 4.5 part 1) that $u_n \to \bar{u}$ strongly in $L^p(\Omega)$. Therefore,

$$u_n \to \bar{u} \quad \text{strongly in} \quad V,$$

as desired. □

Proof of Theorem 1.4. It follows from Proposition 4.1 that the functional I has a nontrivial critical point $u \in K_m$. We will now apply Corollary 4.1 to see that u is nonnegative nontrivial monotonic solution of (2). To do this we need to show there is some $v \in Dom(\psi)$ which satisfies $Av = D\varphi(u)$; or to be more explicit, which satisfies (26). We now prove this. Fix $u \in K_m$ and suppose a satisfies the assumed hypothesis. For $\varepsilon > 0$ small let $u^\varepsilon, a^\varepsilon$ denote the smoothed versions of u and a respectively as promised by Lemma 5.1. Replacing $af(u)$ with $a^\varepsilon f(u^\varepsilon)$ on the right hand side of (26) we can apply Proposition 2.1 to see there is some $v^\varepsilon \in K_m \cap C^{2,\alpha}(\overline{\Omega})$ which satisfies $-\Delta v^\varepsilon + v^\varepsilon = a^\varepsilon f(u^\varepsilon)$ in Ω with $\partial_\nu v^\varepsilon = 0$ on $\partial\Omega$, which has the weak formulation

$$\int_\Omega \nabla v^\varepsilon \cdot \nabla\varphi + v^\varepsilon \varphi dx = \int_\Omega a^\varepsilon f(u^\varepsilon)\varphi dx, \quad \forall \varphi \in H^1_G(\Omega). \tag{51}$$

Note note that we have $|f(u^\varepsilon)| \leq C(1+|u^\varepsilon|^{(p-1)}) \leq C(1+|u|^{(p-1)})$ in Ω and so we have $|f(u^\varepsilon)|^{p'} \leq C_p(1+|u^\varepsilon|^{(p-1)p'}) = C_p((1+|u^\varepsilon|^p)$ in Ω and hence $\|f(u^\varepsilon)\|_{L^{p'}}^{p'} \leq C(1+\|u^\varepsilon\|_{L^p}^p) \leq C(1+\|u\|_{L^p}^p)$.

Taking $\varphi = v^\varepsilon$ in (51) and applying Hölder's inequality on the right hand side one obtains, after using the above bound,

$$\|v^\varepsilon\|_{H^1}^2 \leq \|a^\varepsilon f(u^\varepsilon)\|_{L^{p'}} \|v^\varepsilon\|_{L^p} \leq C\|v^\varepsilon\|_{H^1}$$

where C independent of ε and where we have used the improved Sobolev imbeddings for $v^\varepsilon \in K_m$. This gives an $H^1(\Omega)$ bound on v^ε and hence by passing to a suitable subsequence there is some $v \in H^1_G(\Omega)$ such that $v^\varepsilon \rightharpoonup v$ in $H^1_G(\Omega)$ as $\varepsilon \to 0$. We can then pass to the limit in (51) for all $\varphi \in H^1_G(\Omega) \cap L^\infty(\Omega)$; to pass to the limit on the right hand side we can use the dominated convergence theorem. Noting that K_m is weakly closed in $H^1_G(\Omega)$ we have $v \in K_m$. To complete showing that $v \in Dom(\psi)$ we need to show that $v \in W^{2,q}(\Omega)$ where $q = p'$. Note above that we have shown $a^\varepsilon f(u^\varepsilon)$ is bounded in $L^{p'}(\Omega)$ independently of ε. So apply L^p elliptic regularity theory shows that v^ε is bounded in $W^{2,p'}(\Omega)$. So we now have a nonnegative nonzero sufficiently regularity solution u of (2) and we can then apply the strong maximum principe to see that u is positive.

□

Proof of Corollary 1.5. Suppose Ω is a domain of m revolution which satisfies (3) and that a satisfies A_4 and further we assume that $a(t_1, t_2, ..., t_m) = a(t_1, t_2, ..., t_i)$ for some $1 \leq i < m$. Suppose $A_1 - A_4$ hold with $2 < p < 2_i^*$ in A_2. Applying Theorem 1.4 one sees there is a positive $v = v(t_1, ..., t_i) \in K_i$ which satisfies the lower dimensional problem

$$-\Delta_t v - \sum_{k=1}^{i} \frac{n_k - 1}{t_k} v_{t_k} + v = a(t_1, .., t_i) f(v) \text{ in } Q_i, \qquad \partial_\nu v = 0 \text{ on } \partial Q_i. \tag{52}$$

Set $u(t_1, ..., t_m) := v(t_1, ..., t_i)$ and note that $u \in K_m$ is a nonzero solution

$$-\Delta_t u - \sum_{k=1}^{m} \frac{n_k - 1}{t_k} u_{t_k} + u = a(t_1, ..., t_m) f(u) \text{ in } Q_m, \qquad \partial_\nu u = 0 \text{ on } \partial Q_m. \tag{53}$$

□

5 Appendix

Lemma 5.1. *(Smoothing of $u \in K_m$) Suppose $m \geq 2$, $Q_m = \tilde{\Omega}$ and $u \in K_m$. Then there is some smooth $u^\varepsilon \in K_m$ such that $u^\varepsilon \leq u$ a.e. in Ω and such that $u^\varepsilon \to u$ a.e. in Ω as $\varepsilon \searrow 0$.*

Proof. Consider $0 \leq u \in K_m$ and so $0 \leq u \in Y_m$ with $u_{t_k} \geq 0$ a.e. in Q_m for $1 \leq k \leq m$. Let $0 \leq \eta$ denote a smooth compactly supported function in $(-1, 0)^m$ with $\int_{\mathbb{R}^m} \eta = 1$.

Note that u is defined in Q_m and we then extend u to be zero outside Q_m. Let $\varepsilon > 0$ be small and define $Q^\varepsilon := (-\varepsilon, 0)^m$ and we let $t = (t_1, t_2, .., t_m)$ and $\bar{t} := (\bar{t}_1, ..., \bar{t}_m)$. We then define

$$u^\varepsilon(t) := \int_{Q^\varepsilon} \frac{1}{\varepsilon^m}\eta\left(\frac{\bar{t}}{\varepsilon}\right) u(t+\bar{t})d\bar{t},$$

for $t \in Q_m$. Note that we can re-write u^ε as

$$u^\varepsilon(t) = \int_{\mathbb{R}^m} \frac{1}{\varepsilon^m}\eta\left(\frac{\hat{t}-t}{\varepsilon}\right) u(\hat{t})d\hat{t},$$

and this shows that u^ε is smooth. Let $t \in Q_m$ and returning to the first expression we see

$$u^\varepsilon(t) = \int_{Q^\varepsilon} \frac{1}{\varepsilon^m}\eta\left(\frac{\bar{t}}{\varepsilon}\right) u(t+\bar{t})d\bar{t} \le \int_{Q^\varepsilon} \frac{1}{\varepsilon^m}\eta\left(\frac{\bar{t}}{\varepsilon}\right) u(t)d\bar{t} = u(t),$$

after recalling the support of η and since u has some monotonicity. Now suppose that $t = (t_1, ..., t_m), s = (s_1, ..., s_m) \in Q_m$ with $t_k \le s_k$ for all $1 \le k \le m$. Then note

$$u^\varepsilon(t) = \int_{Q^\varepsilon} \frac{1}{\varepsilon^m}\eta\left(\frac{\bar{t}}{\varepsilon}\right) u(t+\bar{t})d\bar{t} \le \int_{Q^\varepsilon} \frac{1}{\varepsilon^m}\eta\left(\frac{\bar{t}}{\varepsilon}\right) u(s+\bar{t})d\bar{t} = u^\varepsilon(s),$$

and hence we see u^ε has the desired monotonicity. One can then use a the standard approach to show that $u^\varepsilon \to u$ a.e. in Q_m as $\varepsilon \searrow 0$.

□

Proof of Proposition 2.1. Assume Ω is domain of m revolution in $\mathbb{R}^N$ which satisfies (3) and fix $0 \le h \in K_m \cap C^{1,1}(\overline{\Omega})$. As before we identify $H^1_G(\Omega)$ and Y_m and hence, by standard arguments there is a unique $v \in Y_m$ which satisfies

$$\int_{Q_m} (\nabla_t v \cdot \nabla_t \eta + v\eta)\, d\mu_m(t) = \int_{Q_m} h(t)\eta\, d\mu_m(t), \quad \forall \eta \in Y_m. \tag{54}$$

Our goal now is to show that $v \in K_m \cap C^{2,\alpha}(\overline{\Omega})$ and we first show that $v \in K_m$. To do that we begin by solving a smoothed version of (54). For $\varepsilon > 0$ small define $d\mu^\varepsilon_m(t) = \prod_{i=1}^m (t_i + \varepsilon)^{n_i - 1} dt_i$ and consider the energy

$$E_\varepsilon(v) := \frac{1}{2}\int_{Q_m} \left(|\nabla_t v|^2 + v^2 - h(t)v\right) d\mu^\varepsilon_m(t).$$

It is easily seen that E_ε attains its minimum on $H^1(Q_m) \subset Y_m$ at v^ε which satisfies

$$\begin{cases} -\Delta_t v^\varepsilon - \sum_{k=1}^m \frac{n_k - 1}{t_k + \varepsilon} v^\varepsilon_{t_k} + v^\varepsilon = h & \text{in } Q_m, \\ \partial_\nu v^\varepsilon = 0 & \text{on } \partial Q_m. \end{cases} \tag{55}$$

Note now that the solution of this problem is as smooth as the right hand side and the nonsmooth domain allow. We now proceed in several steps.

Step 1. For $0 < \alpha < 1$ and $0 < \varepsilon < 1$ one has $v^\varepsilon \in C^{2,\alpha}(\overline{Q_m})$. We prove this result in the case of a double domain of revolution; but it extends to the general case. We will use the method of even reflections to prove the global regularity result. Towards this define $c_k^\varepsilon(t) = \frac{n_k-1}{t_k+\varepsilon}$ and let $c^\varepsilon(t)$ be the 2 dimensional vector with components $c_1^\varepsilon(t)$ and $c_2^\varepsilon(t)$. Define the even extension of v^ε by

$$\overline{v^\varepsilon}(t_1,t_2) := \begin{cases} v^\varepsilon(t_1,t_2) & \text{in } Q_2 \\ v^\varepsilon(-t_1,t_2) & \text{in } (-1,0)\times(0,1) \\ v^\varepsilon(-t_1,-t_2) & \text{in } (-1,0)\times(-1,0) \\ v^\varepsilon(t_1,-t_t) & \text{in } (0,1)\times(-1,0) \end{cases}$$

and where $\overline{v^\varepsilon}$ is extended to the axis by continuity. We now define $\overline{c_k^\varepsilon}(t)$ to be the odd reflection of c_k^ε across $t_k = 0$, ie. given by

$$\overline{c_k^\varepsilon}(t) := \begin{cases} c_k^\varepsilon(t) & \text{in } t_k \geq 0 \\ -c_k^\varepsilon(-t) & \text{in } t_k < 0, \end{cases}$$

and note that $\overline{c_k^\varepsilon}$ has a jump discontinuity at $t_k = 0$. Set $\overline{c^\varepsilon}$ to be the 2 dimensional vector with components $\overline{c_k^\varepsilon}(t)$.

We then let $\overline{h}$ denote the even extension of h as we did with v^ε. Consider the nested cubes

$$D_4 \subset D_3 \subset D_2 \subset D_1 \subset (-1,1)\times(-1,1)\,,$$

where each is compactly contained in the other. We then have $\overline{v^\varepsilon}$ is a weak solution of

$$-\Delta_t \overline{v^\varepsilon} + \overline{v^\varepsilon} = \overline{h} + \overline{c^\varepsilon}(t)\cdot\nabla_t \overline{v^\varepsilon} \quad \text{in } D_1. \tag{56}$$

Note that since $v^\varepsilon \in H^1(Q_2)$ we have $\overline{v^\varepsilon} \in H^1(D_1)$ which implies the right hand side of (56) belongs to $L^2(D_1)$.

We can then apply elliptic regularity to see that $\overline{v} \in H^2(D_2)$ and since we are in dimension 2 we can apply the Sobolev imbedding theorem to see that $\overline{v^\varepsilon}_{t_k} \in L^q(D_2)$ for all $1 \leq q < \infty$. In the case of $m > 2$ one can apply a bootstrap argument to obtain $\overline{v^\varepsilon}_{t_k} \in L^q(D_2)$ for all $1 \leq q < \infty$. We can then apply L^q elliptic regularity theory and the Sobolev imbedding theorem to see that $\overline{v^\varepsilon} \in C^{1,\alpha}(D_3)$ for all $0 < \alpha < 1$. We now show the right hand side of (56) is Hölder continuous on D_3.

Claim 1. $(\overline{c^\varepsilon}\cdot\nabla_t\overline{v^\varepsilon}) \in C^{0,\alpha}(D_3)$. This is not immediately obvious since $\overline{c^\varepsilon}_k$ has jump discontinuities. We now show that $\overline{c^\varepsilon}_1\overline{v^\varepsilon}_{t_1} \in C^{0,\alpha}(D_3)$.
Consider $(t_1,t_2),(\tau_1,\tau_2) \in D_3$ and consider the three cases:
(i) $t_1 < 0$ and $\tau_1 \geq 0$, (ii) $t_1,\tau_1 \geq 0$, (iii) $t_1,\tau_1 < 0$.

Case (i). Now note that

$$\left|(\overline{c^\varepsilon}_1\overline{v^\varepsilon}_{t_1})(t_1,t_2) - (\overline{c^\varepsilon}_1\overline{v^\varepsilon}_{t_1})(\tau_1,\tau_2)\right| \leq C\left(\left|\overline{v}_{t_1}(t_1,t_2)\right| + \left|\overline{v^\varepsilon}_{t_1}(\tau_1,\tau_2)\right|\right),$$

and using the the fact that $v_{t_1}^\varepsilon = 0$ on $t_1 = 0$ and $\overline{v^\varepsilon}_{t_1} \in C^{0,\alpha}(D_3)$ we have

$$\left|\overline{v^\varepsilon}_{t_1}(t_1,t_2)\right| = \left|\overline{v^\varepsilon}_{t_1}(t_1,t_2) - \overline{v^\varepsilon}_{t_1}(0,t_2)\right| \leq C|t_1|^\alpha,$$

and similarly we have $|\overline{v^\varepsilon}_{t_1}(\tau_1,\tau_2)| \le C|\tau_1|^\alpha$. From this we see

$$\frac{\big|(\overline{c^\varepsilon}_1\overline{v^\varepsilon}_{t_1})(t_1,t_2)-(\overline{c}_1\overline{v^\varepsilon}_{t_1})(\tau_1,\tau_2)\big|}{\big|(t_1,t_2)-(\tau_1,\tau_2)\big|^\alpha} \le \frac{C|t_1|^\alpha + C|\tau_1|^\alpha}{\big|(t_1-\tau_1,t_2-\tau_2)\big|^\alpha},$$

and note the right hand side is bounded after recalling that $t_1<0$ and $\tau_1 \ge 0$.

For case (ii) and (iii) we easily see the needed Hölder quotient is bounded after we consider that $\overline{c_1^\varepsilon}$ is smooth on the restricted domain and since $\overline{v^\varepsilon}_{t_1}$ is Hölder continuous. This completes the proof of the claim.

Using the above claim we now see that the right hand side of (56) is in $C^{0,\alpha}(D_3)$ and hence we can now apply elliptic regularity theory to see that $\overline{v^\varepsilon} \in C^{2,\alpha}(\overline{D_4})$. We now argue that one in fact has $v^\varepsilon \in C^{2,\alpha}(\overline{Q_2})$. We could extend v^ε evenly across the outer boundaries and the extension would satisfy a similar equation as above. Note one difference now is that $\overline{h}$ is sufficiently smooth and symmetric across $t_k=0$ and hence its even extension is sufficiently regular. When we extend h across the outer boundary it will, in general, only be Lipschitz continuous. But this is enough to carry out the above procedure. Carrying this out gives $v^\varepsilon \in C^{2,\alpha}(\overline{Q_2})$, which complete the proof of Step 1.

Step 2. We now show that $v^\varepsilon \in K_m$. We do this step for all $m \ge 2$. First note that we have $v^\varepsilon \ge 0$ and so we need only show that v^ε has the desired monotonicity; $v^\varepsilon_{t_k} \ge 0$ in Q_m for $1 \le k \le m$.

Consider $w^\varepsilon := v^\varepsilon_{t_1}$ and note that $w^\varepsilon \in C^{1,\alpha}(\overline{\Omega_m})$ is a weak solution of

$$\begin{cases} -\Delta_t w^\varepsilon - c^\varepsilon(t)\cdot\nabla_t w^\varepsilon + \left(1+\frac{n_1-1}{(t_1+\varepsilon)^2}\right) w^\varepsilon &= h_{t_1} \qquad \text{in } Q_m, \\ w^\varepsilon &= 0 \qquad \text{on } \partial_1 Q_m, \\ w^\varepsilon_{t_i} &= 0 \qquad \text{on } \partial_i Q_m, \quad \text{for } 2 \le i \le m \end{cases} \tag{57}$$

where $\partial_i Q_m := \{t \in \partial Q_m : t_i \in \{0,1\}$ and for $\forall j \ne i$ we have $t_j \notin \{0,1\}\}$. Note there are no issues regarding the boundary conditions since one has enough regularity to pass the required derivatives on the boundary. Note that the right hand side of (57) is nonnegative. Note a weak solution w^ε of (57) satisfies

$$\int_{Q_m}\left(\nabla_t w^\varepsilon\cdot\nabla_t\eta + \left\{1+\frac{n_1-1}{(t_1+\varepsilon)^2}\right\} w^\varepsilon\eta\right) d\mu_m^\varepsilon(t) = \int_{Q_m} h_{t_1}\eta\, d\mu_m^\varepsilon(t), \forall \eta \in X, \tag{58}$$

where $h_{t_1} \ge 0$ in Q_m and where $X := \{\eta \in H^1(Q_m) : \eta = 0 \text{ on } \partial_1 Q_m\}$. In particular we can take $\eta = (w^\varepsilon)^- \in X$ (the negative part of w^ε to see that

$$\int_{Q_m}\left(|\nabla_t (w^\varepsilon)^-|^2 + \left\{1+\frac{n_1-1}{(t_1+\varepsilon)^2}\right\}((w^\varepsilon)^-)^2\right) d\mu_m^\varepsilon(t) = -\int_{Q_m} h_{t_1}(w^\varepsilon)^- d\mu_m^\varepsilon(t) \le 0$$

and hence $(w^\varepsilon)^- = 0$ a.e. in Q_m. After doing this for each $1 \le k \le m$ we can conclude that $v^\varepsilon \in K_m$.

Step 3. We now need to send $\varepsilon \searrow 0$. First recall that v^ε satisfies (55) and whose weak formulation is given by

$$\int_{Q_m} \left(\nabla_t v^\varepsilon \cdot \nabla_t \eta + v^\varepsilon \eta\right) d\mu_m^\varepsilon(t) = \int_{Q_m} h\eta d\mu_m^\varepsilon(t), \forall \eta \in H^1(Q_m). \tag{59}$$

Taking $\eta = v^\varepsilon$ one easily obtains a bound on $\{v^\varepsilon\}_{0<\varepsilon\leq 1}$ in Y_m and hence after passing to a suitable subsequence one can assume that $v^\varepsilon \to \tilde{v}$ weakly in Y_m and $\tilde{v}$ satisfies (54) with v replaced with $\tilde{v}$. But recalling v is already a solution of (54) and since the solution is unique we see that $v = \tilde{v} \in K_m$ after noting that since K_m is convex and closed in Y_m and hence it is weakly closed.

We now show that v has added regularity. To do this we obtains bounds on $\{v^\varepsilon\}_\varepsilon$ independent of $0 < \varepsilon < 1$. Set $\widehat{w}^\varepsilon(t) := \frac{w^\varepsilon(t)}{t_1+\varepsilon} = \frac{v^\varepsilon_{t_1}(t)}{t_1+\varepsilon}$ and using (57) we see that $\widehat{w}^\varepsilon \in C^{1,\alpha}(\overline{Q_m})$ is a weak solution of

$$\left\{ \begin{array}{rcll} -\Delta_t \widehat{w}^\varepsilon - b^\varepsilon(t) \cdot \nabla_t \widehat{w}^\varepsilon + \widehat{w}^\varepsilon & = & \frac{h_{t_1}}{t_1+\varepsilon} & \text{in } Q_m, \\ \widehat{w}^\varepsilon & = & 0 & \text{on } \partial_1 Q_m, \\ \widehat{w}^\varepsilon_{t_i} & = & 0 & \text{on } \partial_i Q_m, \quad \text{for } 2 \leq i \leq m \end{array} \right. \tag{60}$$

where $b^\varepsilon(t) = \big(b_1^\varepsilon(t), ..., b_m^\varepsilon(t)\big)$ with $b_k^\varepsilon(t) := \frac{n_k - 1 + \alpha_k}{t_k+\varepsilon}$ where $\alpha_1 = 2$ and $\alpha_k = 0$ for $2 \leq k \leq m$. Using suitable reflections one can again show that $\widehat{w}^\varepsilon \in C^{2,\alpha}(\overline{Q_m})$ for each fixed $0 < \varepsilon < 1$. Note that we have $0 \leq \widehat{w}^\varepsilon$ and we now proceed to show that $\widehat{w}^\varepsilon$ is bounded independently of $0 < \varepsilon < 1$. So let $t^0 = (t_1^0, ..., t_m^0) \in \overline{Q_m}$ be such that $\widehat{w}^\varepsilon(t^0) = \sup_{Q_m} \widehat{w}^\varepsilon$ and we can assume $\widehat{w}^\varepsilon(t^0) > 0$. If $t^0 \in Q_m$ then we have $\Delta_t w^\varepsilon(t^0) \leq 0$. Now suppose that $t^0 \in \partial Q_m$. Note from the boundary condition we have $t^0 \notin \overline{\partial_1 Q_m}$. Set $I := \{1 \leq i \leq m : \exists \delta > 0 \text{ such that } t^0 + \tau e_i \in \overline{Q_m} \text{ for } |\tau| < \delta \}$ where e_i is the i^{th} standard basis vector in $\mathbb{R}^m$ and let $J := \{2, 3, ..., m\} \backslash I$; note that 1 is always an element of I. Then note that since $\widehat{w}^\varepsilon$ has a maximum at t^0 we have $\widehat{w}^\varepsilon_{t_i}(t^0) = 0$ and $\widehat{w}^\varepsilon_{t_i t_i}(t^0) \leq 0$ for all $i \in I$; note here we are just using single variable calculus at a interior maximum point. Now note that for $i \in J$ we have $t^0 \in \overline{\partial_i Q_m}$ and hence we have $\widehat{w}^\varepsilon_{t_i}(t^0) = 0$ after considering the boundary condition. Using the fact that $w^\varepsilon(t_0)$ has a maximum at t^0 and since $\widehat{w}^\varepsilon_{t_i}(t^0) = 0$ we must have $\widehat{w}^\varepsilon_{t_i t_i}(t^0) \leq 0$. From this we obtain that

$$\begin{array}{rcl} \widehat{w}^\varepsilon(t^0) & = & \Delta_t \widehat{w}^\varepsilon(t^0) + b^\varepsilon(t^0) \cdot \nabla_t \widehat{w}^\varepsilon(t^0) + \dfrac{h_{t_1}(t^0)}{t_1^0 + \varepsilon} \\ & \leq & \dfrac{h_{t_1}(t^0)}{t_1^0 + \varepsilon} \\ & = & \dfrac{\big(h_{t_1}(t^0) - h_{t_1}(\overline{t})\big)}{|t^0 - \overline{t}|} \dfrac{|t^0 - \overline{t}|}{t_1^0 + \varepsilon} \quad \text{where } \overline{t} = (0, t_2^0, ..., t_m^0) \\ & \leq & \dfrac{\|h\|_{C^{1,1}} |t_1^0|}{t_1^0 + \varepsilon} \leq \|h\|_{C^{1,1}}, \end{array}$$

and so we have shown $0 \leq \frac{v^\varepsilon_{t_k}(t)}{t_k+\varepsilon} \leq \|h\|_{C^{1,1}}$ in Q_m for $k = 1$ and, by using the same argument, it also holds for all $1 \leq k \leq m$. We now return to (56) which we recall was

$-\Delta_t \overline{v^\varepsilon} + \overline{v^\varepsilon} = \overline{h} + \overline{c^\varepsilon}(t) \cdot \nabla_t \overline{v^\varepsilon}$ in D_1. Note that $\overline{c^\varepsilon}(t) \cdot \nabla_t \overline{v^\varepsilon}$ is bounded independently of ε after considering the bound on $\frac{v^\varepsilon_{t_k}}{t_k+\varepsilon}$. We can then apply elliptic regularity to see that $\overline{v^\varepsilon}$ is bounded in $C^{1,\alpha}(D_2)$ (independently of ε) for $0 < \alpha < 1$. We can now argue as before to show that $\overline{c^\varepsilon} \cdot \nabla_t \overline{v^\varepsilon}$ is bounded in $C^{0,\alpha}(D_2)$ independently of ε and hence we can apply elliptic regularity theory to see that $\overline{v^\varepsilon}$ is bounded in $C^{2,\alpha}(D_3)$ independently of ε. To obtain global regularity we need to perform the even extension across the outer boundaries. Note, as mentioned before, the even extension of h will now only by Lipschitz, but this is sufficient to show that v^ε is bounded in $C^{2,\alpha}(\overline{Q_m})$ independently of ε and this gives us the desired result.

□

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