Singular solutions of elliptic equations on a perturbed cone

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Abstract

In this work we obtain positive singular solutions of

$$\begin{cases} -\Delta u(y) = u(y)^p & \text{in } y \in \Omega_t, \\ u = 0 & \text{on } y \in \partial\Omega_t, \end{cases}$$

where Ω_t is a sufficiently small $C^{2,\alpha}$ perturbation of the cone $\Omega := \{x \in \mathbb{R}^N : x = r\theta, r > 0, \theta \in S\}$ where $S \subset S^{N-1}$ has a smooth nonempty boundary and where $p > 1$ satisfies suitable conditions. By singular solution we mean the solution is singular at the 'vertex of the perturbed cone'. We also consider some other perturbations of the equation on the unperturbed cone Ω and here we use a different class of function spaces.

Contents

1 Introduction

In this work we consider a Lane-Emden-Fowler equation on small perturbations of a cone in $\mathbb{R}^N$. To describe the cone we fix $S \subset S^{N-1}$ $(N \geq 3)$ with a smooth nonempty boundary and we now consider the cone given by $\Omega := \{x \in \mathbb{R}^N : r = |x| > 0, \theta = \frac{x}{|x|} \in S\}$. We are interested in obtaining positive singular solutions of

$$\begin{cases} -\Delta_y u(y) &= u(y)^p \qquad \text{in } \Omega_t, \\ u &= 0 \qquad \text{on } \partial\Omega_t, \end{cases} \tag{1}$$

where Ω_t is a sufficiently small perturbation of Ω and where $p > 1$.

The approach we take is to linearize around a positive separable solution v_0 to the unperturbed problem given by

$$\begin{cases} -\Delta v_0 &= v_0^p \qquad \text{in } \Omega, \\ v_0 &= 0 \qquad \text{on } \partial\Omega \backslash \{0\}. \end{cases} \tag{2}$$

By a well known computation it is known that if w is a positive classical solution of

$$\begin{cases} -\Delta_\theta w &= \nu\,(N-2+\nu)\,w + w^p \qquad \text{in } S, \\ w &= 0 \qquad \text{on } \partial S, \end{cases} \tag{3}$$

where $\nu := \frac{-2}{p-1}$ and Δ_θ is the Laplace-Beltrami operator on S^{N-1}, then $v_0(x) = v_0(r,\theta) = r^\nu w(\theta)$ is positive singular solution of (2). When we need to indicate the dependence of w on p we will write $w = w_p$ and from here on we shall define

$$\lambda_p := \nu\,(N-2+\nu) = \frac{2}{p-1}\left(\frac{2}{p-1} - (N-2)\right).$$

We now discuss the restrictions on p so we can find a positive classical solution of (3). For the existence we use a variational approach; we minimize

$$E_p(w) := \frac{\int_S |\nabla_\theta w|^2 - \lambda_p w^2 d\theta}{\left(\int_S |w|^{p+1} d\theta\right)^{\frac{2}{p+1}}},$$

over $H_0^1(S)\backslash\{0\}$. We now introduce the following critical values of p;

$$p_0 := 1 + \frac{4}{N-2+\sqrt{(N-2)^2+4\lambda_1(S)}}, \quad p_1 := \frac{N+3}{N-1}, \quad p_2 := \frac{N+1}{N-3}, \tag{4}$$

where we are using the obvious generalizations in the case of $N = 3$. Note p_2 is exactly the exponent coming from the critical Sobolev imbedding in dimension $N-1$, when $N > 3$.

- For $p \in (p_0, p_2)$ one has $\lambda_p < \lambda_1(S)$ (here $\lambda_1(S)$ is the first eigenvalue of $-\Delta_\theta$ in $H_0^1(S)$) and $H_0^1(S) \subset\subset L^{p+1}(S)$. For this range of p the above variational approach coupled with an elliptic regularity argument shows the existence of a positive classical solution w of (3).

- For $p \le p_0$ one has $\lambda_p \ge \lambda_1(S)$ and hence there is no positive classical solution of (3).

- For $p \ge p_2$ note that (3) becomes a critical/supercritical problem in the sense of Sobolev imbedding and the existence of positive classical solutions of (3) becomes a very hard question which we won't discuss, other than to mention for certain symmetrical domains one can find a positive solution; for instance a geodesic annulus.

In our approach we will require that not only is w a positive classical solution of (3) but we will also need it to be a nondegenerate solution; by this we mean the associated linearized operator has a trivial kernel. In the case of S a geodesic ball of radius α, where $0 < \alpha \le \frac{\pi}{2}$, then w is nondegenerate for $p \in (p_0, p_1)$ which follows from Theorem A [38], see section 1.2. For the range of $p \in (p_1, p_2)$ they also show the solutions are nondegenerate in the space of radial functions, but this is not sufficient for our purposes.

With this in mind we will apply some abstract analytic bifurcation theory developed in [12, 5, 4] to show the existence of at most a countable sequence of p's (they may be none or just a finite sequence) increasing to p_2 and for which w_p is a nondegenerate solution provided p is not an element of the sequence. In the case of general domains S we will apply the abstract bifurcation theory to show w_p is a nondegenerate except again for a countable increasing sequence; but now this sequence is contained in (p_0, p_2) with p_2 being the only possible limit point.

We now discuss the perturbations we will consider and also the change of variables to reduce the problem to one on the unperturbed cone; we mention the following change of variables is taken from [16] where they examine the extremal solution of the Gelfand problem on perturbations of the unit ball. Let $\psi : \overline{\Omega} \to \mathbb{R}^N$ be $C^{2,\alpha}$ map such that there is some $C > 0$ such that

$$|\psi(x)| \le C|x|, \qquad |D\psi(x)| \le C, \qquad |D^2\psi(x)| \le \frac{C}{|x|}. \tag{5}$$

For $t > 0$ small we now define the perturbed domain

$$\Omega_t := \{y : y = x + t\psi(x), x \in \Omega\}.$$

Given $y \in \Omega_t$ define $\tilde{\psi}(t, y)$ via $x = y + t\tilde{\psi}(t, y)$. Then there is some $C_1 > 0$ such that for small enough $t > 0$ one has

$$|\tilde{\psi}(t, y)| \le C_1|y|, \qquad |D_y\tilde{\psi}(t, y)| \le C_1, \qquad |D_y^2\tilde{\psi}(t, y)| \le \frac{C_1}{|y|}. \tag{6}$$

Given $u(y)$ defined on Ω_t we define $v(x)$ on Ω via $u(y) = v(x)$ where y and x are related as above. To find a positive solution of (1) it is sufficient to find a positive solution $v(x)$ of

$$\begin{cases} -\Delta v - E_t(v) = v^p & \text{in } x \in \Omega, \\ v = 0 & \text{on } \partial\Omega, \end{cases} \tag{7}$$

where E_t is the second order linear differential operator given by

$$E_t(v) := 2t \sum_{i,k} v_{x_i x_k} \partial_{y_i} \tilde{\psi}_k + t \sum_{i,k} v_{x_k} \partial_{y_i y_i} \tilde{\psi}_k + t^2 \sum_{i,j,k} v_{x_j x_k} \partial_{y_i} \tilde{\psi}_j \partial_{y_i} \tilde{\psi}_k,$$

where in all the sums the indices run from 1 to N.

We now state our main result.

Theorem 1. *1. Suppose $N \geq 3$ and $S \subset S^{N-1}$ with smooth nonempty boundary. Then there is a sequence of increasing q_k (possibly empty or finite) with $p_0 < q_1 < q_2 < \ldots$ with $q_k < p_2$ (with p_2 being the only possible limit point) such that for all $p \in (p_0, p_2) \setminus \left(\{q_k : k \geq 1\} \cup \{\frac{N+2}{N-2}\} \right)$ and all mappings ψ which satisfy (5) there exists a positive singular solution of (1) for sufficiently small $t > 0$.*

2. Suppose $N \geq 3$ and $S \subset S^{N-1}$ is a geodesic ball. Then there is a sequence of increasing q_k (possibly empty or finite) with $p_1 \leq q_1 < q_2 < \ldots$ with $q_k < p_2$ (with p_2 being the only possible limit point) such that for all $p \in (p_0, p_2) \setminus \left(\{q_k : k \geq 1\} \cup \{\frac{N+2}{N-2}\} \right)$ and all mappings ψ which satisfy (5) there exists a positive singular solution of (1) for sufficiently small $t > 0$.

Our approach to finding a positive solution of (7), for small t, will be to linearize around v_0 where v_0 is an explicit singular separable solution as defined above; for the time being we are assuming the existence of a classical positive solution $w = w_p$ of (3). Of course a crucial ingredient in this approach will be the mapping properties of the linearized operator associated with v_0; ie.

$$L(\phi)(x) := -\Delta \phi(x) - p v_0^{p-1}(x) \phi(x) = -\Delta \phi(x) - \frac{p w_p(\theta)^{p-1}}{r^2} \phi(x). \tag{8}$$

We now look for solutions of (7) of the form $v(x) = v_0(x) + \phi(x)$ where ϕ is to be determined. Then we need ϕ to satisfy

$$\begin{cases} L(\phi) = (v_0 + \phi)^p - v_0^p - p v_0^{p-1} \phi + E_t(v_0) + E_t(\phi) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases} \tag{9}$$

To find a solution of this we will apply Banach's fixed point theorem on a suitable space to the nonlinear operator given by $J_t(\phi) = \psi$ where ψ satisfies

$$\begin{cases} L(\psi) = (v_0 + \phi)^p - v_0^p - p v_0^{p-1} \phi + E_t(v_0) + E_t(\phi) & \text{in } \Omega, \\ \psi = 0 & \text{on } \partial\Omega. \end{cases} \tag{10}$$

Note a priori that the term $(v_0 + \phi)^p$ is not well defined if $v_0 + \phi$ is negative somewhere in Ω; this won't be an issue since we will restrict ϕ to be small enough such that we always have this term is positive.

For this approach to work one will need to understand the the mapping properties of L and in particular we will want L to be surjective. We now define the functions spaces we will work in, these spaces are a very slight adjustment of the spaces introduced in [32, 34].

Let $0 < \alpha < 1$ be fixed and we consider the following function spaces, where $A_s := \{x \in \Omega : s < |x| < 2s\}$,

$$\|f\|_{C^{0,\alpha}_{\nu-2}} = \sup_{0<s} s^{2-\nu} \left(\sup_{A_s} |f| + s^\alpha \sup_{x,y\in A_s} \frac{|f(x)-f(y)|}{|x-y|^\alpha} \right),$$

$$\|\phi\|_{C^{2,\alpha}_\nu} = \sup_{0<s} s^{-\nu} \left\{ \sup_{A_s} |\phi| + s \sup_{A_s} |\nabla\phi| + s^2 \sup_{A_s} |D^2\phi| + s^{2+\alpha} \sup_{x,y\in A_s} \frac{|D^2\phi(x)-D^2\phi(y)|}{|x-y|^\alpha} \right\}.$$

Set $Y := C^{0,\alpha}_{\nu-2}$ and X to be the set of $\phi \in C^{2,\alpha}_\nu$ with $\phi = 0$ on $\partial\Omega\backslash\{0\}$; with given norms $\|\cdot\|_Y$ and $\|\cdot\|_X$.

1.0.1 Brief outline of paper

As mentioned above, a key point in showing the mapping L is onto Y, will be that w is a nondegenerate solution of (3). Assuming the nondegeneracy condition on w we analyse in detail the linearized operator L in Section 2. In Section 1.2 we state known results regarding w in the case of S a geodesic ball, and in Section 3 we consider the case of general S. We then perform the fixed point argument in Section 4. In Section 5 we consider some more general equations using the same method; and in particular the same function spaces. We then conclude the paper with Section 6 where we examine the equations in different function spaces. The purpose of this is to allow for larger perturbations of the unperturbed cone. Instead of working directly with perturbations of the domain we prefer to just consider lower order perturbations of the domain to illustrate the usefulness of the new spaces.

1.1 General background

A well studied problem is the existence versus non-existence of positive solutions of the Lane-Emden equation given by

$$\begin{cases} -\Delta u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{11}$$

where Ω is a bounded domain in $\mathbb{R}^N$ with $N \geq 3$. Define the critical exponent $p_s = \frac{N+2}{N-2}$ and note that it is related to the critical Sobolev imbedding exponent $2^* := \frac{2N}{N-2} = p_s + 1$. For $1 < p < p_s$ $H^1_0(\Omega)$ is compactly imbedded in $L^{p+1}(\Omega)$ and hence standard methods show the existence of a positive minimizer of

$$\min_{u\in H^1_0(\Omega)\backslash\{0\}} \frac{\int_\Omega |\nabla u|^2 dx}{\left(\int_\Omega |u|^{p+1} dx\right)^{\frac{2}{p+1}}}.$$

This positive minimizer is a positive solution of (11) see for instance the book [39]. For $p \geq p_s$ $H_0^1(\Omega)$ is no longer compactly imbedded in $L^{p+1}(\Omega)$ and so to find positive solutions of (11) one needs to take other approach. For $p \geq p_s$ the well known Pohozaev identity [36] shows there are no positive solutions of (11) provided Ω is star shaped. For general domains in the critical/supercritical case, $p \geq p_s$, the existence versus nonexistence of positive solutions of (11) is a very delicate question; see [9, 35, 20, 18, 19] and for related problems [17, 28, 8, 27, 41].

There has been much work done on the existence and nonexistence of positive classical solutions of

$$-\Delta w = w^p \qquad \text{in } \mathbb{R}^N. \tag{12}$$

As in the bounded domain case the critical exponent p_s plays a crucial role. For $1 < p < p_s$ there are no positive classical solutions of (12) and for $p \geq p_s$ there exist positive classical solutions, see [6, 7, 25, 24]. The moving plane method shows that all positive classical solutions, satisfying certain assumptions, are radial about a point. Regarding the existence versus nonexistence of stable solutions of (12) one should consult [40, 29, 23]. For the existence of solutions for (12) in either exterior domains; or perturbations of (12) see [13, 14, 15]. The Dirichlet problem (11) in the context of very weak solutions (and which allows for singularities on the boundary) have been studied in [21, 26, 30, 33]. We now briefly mention here some works relevant to our study. In [21], del Pino-Musso-Pacard constructed positive weak solutions of the problem (11) which vanish in suitable trace sense on $\partial\Omega$, but which are singular at prescribed single points if p is equal or slightly above $\frac{N+1}{N-1}$ (they also consider the case for a different range of p where the singular set is higher dimensional). Also, when $\Omega = \mathbb{R}^N_+$ they constructed a solution of problem with fast decay, behaves like as $|x|^{-(N-1)}$ as $|x| \to \infty$. In [30] Horák-McKennab-Reichel considered the equation (11) in Lipschitz wedge- like domains Ω, smooth domains except for one corner, where it locally coincides with a cone of cross-section $S \subset S^{N-1}$ (see [26, 30] for the precise definition). They proved the existence of an unbounded, positive, very weak solution which blows up at $0 \in \partial\Omega$, and when Ω is an infinite cone then the equation admits a positive solution behaves like $|x|^{\frac{-2}{p-1}}$ fro any $p \in (p_0, \infty)$ if $N = 2, 3$ and any $p \in (p_0, \frac{N+1}{N-3})$ if $N \geq 4$, (p_0 defined in (4)) and note that p_0 depends on S. But this solution does not have fast decay at infinity. Note that the exponent p_0 is a truly critical exponent as it is shown in [33] that if $1 < p < p_0$ then every very weak solution of problem (11) is bounded. Very recently, Konstantinos T. Gkikas in [26] improved some of the above results and based on a fixed point argument which also allows the construction of blowing-up solutions, he showed the existence of positive weak solutions which vanish in a suitable trace sense on ∂S, but which are singular at prescribed "edge" of Ω if p is equal or slightly above the exponent p_0. Moreover, in the case which Ω is unbounded, the solutions have fast decay at infinity like $|x|^{2-\gamma-N}$, where $\gamma := \frac{2-N}{2} + \sqrt{(\frac{N-2}{2})^2 + \lambda_1(S)}$. Note that $2 - \gamma - N = \frac{-2}{p_0-1} < \frac{-2}{p-1}$ for all $p > p_0$, for precise statements and further results see Theorems 1.1-1.3 in [26].

However, the existence of unbounded very weak solutions of (11) for all exponents above the critical point $\frac{N+1}{N-1}$ is still open both in case of smooth domains and domains with conical corners. In this paper, using the idea of perturbing an explicitly known singular solution and then utilizing fixed point argument, we show the existence of positive singular solutions in domains which are small perturbations of the cone, for $p \in (p_0, \frac{N+1}{N-3})$ (see Theorems 1) except for at most countably many values of p. This countable set is coming from the fact we are applying some abstract analytic bifurcation theory and real analyticity methods developed in [12, 5] to obtain a nondegenerate solution of (3) (see Section 3) which in turn allows us to develop the needed linear theory (see Section 2) so as to apply a fixed point argument (see Section 4) to solve (9). It would be interesting to discover whether there really is a resonance phenomena or whether this countable set of p is only an artifact of our proof. Another interesting question would whether one can use a similar approach to investigate the case of a bounded domain with conic singularity.

Before giving more background on problems on the cone we mention that our current work is heavily motivated by [16, 32, 34, 13, 14, 15]. For explicit results on cone domains see [1, 2, 37, 3] and the references within.

1.2 Known results regarding (3) when S the geodesic ball

We will now give a result from [38] but they perform a change of variables so the resulting domain is Euclidean (which we also do); what follows is all taken from [38]. Suppose S is the geodesic ball of radius α, $0 < \alpha \leq \frac{\pi}{2}$ and $P : S^{N-1} \setminus (0, ..., 0, -1) \mapsto R^{N-1}$ is the stereographic projection and set

$$R = \tan \frac{\alpha}{2} \quad \textit{and} \quad B_R = \{x \in R^{N-1} : \ |x| < R\} = P(S).$$

Then w is a positive classical solution of (3) if and only if the function v defined by

$$v(x) = \frac{w(P^{-1}x)}{(1+|x|^2)^{\frac{N-3}{2}}} \quad x \in \overline{B_R}, \tag{13}$$

is a positive classical solution of

$$\begin{cases} \Delta v + \frac{(N-1)(N-3)+4\lambda_p}{(1+|x|^2)^2} v + 4(1+|x|^2)^{\frac{p(N-3)-(N+1)}{2}} v^p & = & 0 & \text{in } B_R, \\ v & = & 0 & \text{on } \partial B_R. \end{cases} \tag{14}$$

They then prove the following theorem.

Theorem A. *[38] If*

$$1 + \frac{4}{N-2+\sqrt{(N-2)^2+4\lambda_1(S)}} < p < \frac{N+1}{N-3} \quad \textit{when} \quad N \geq 3, \tag{15}$$

then problem (3) has a unique positive radial solution w. Moreover, let v be the positive radial solution to (14) defined by (13). Then v is nondegenerate in the radial function space $H^1_{0,rad}(B_R)$.
Moreover, if

$$1 + \frac{4}{N-2+\sqrt{(N-2)^2+4\lambda_1(S)}} < p < \frac{N+3}{N-1} \quad \textit{when} \quad N \geq 3, \tag{16}$$

then v is nondegenerate in the full space $H_0^1(B_R)$.

First we note the exponents defined earlier p_0, p_1, p_2 are coming from this theorem. Note the nondegeneracy conditions carry over to (3). In particular we see w is a nondegenerate solution of (3) in $H_0^1(S)$ for $p \in (p_0, p_1)$; and nondegenerate in $H^1_{0,rad}(S)$ for $p \in [p_1, p_2)$.

2 The linear theory

In this section we consider the needed linear theory so as to apply a fixed point argument (see Section 4) to solve (9). In particular we will want the linearized operator L (around the solution v_0) to be onto Y with a continuous right inverse. Let $w = w_p$ denote a positive classical solution of (3) and consider the eigenpairs

$$-\Delta_\theta \psi_k(\theta) - \nu(N-2+\nu)\psi_k(\theta) - pw_p(\theta)^{p-1}\psi_k(\theta) = \mu_k \psi_k(\theta) \quad \text{in } S, \tag{17}$$

with $\psi_k = 0$ on ∂S and we assume these are $L^2(S)$ normalized.

Lemma 1. *(Kernel of L) Let $p \in (p_0, p_2)$ with $p \neq \frac{N+2}{N-2}$ and suppose $w > 0$ a smooth solution of (3) and we suppose $\phi \in X$ is such that $L(\phi) = 0$ in Ω with $\phi = 0$ on $\partial\Omega$. If $\mu_k \neq 0$ for all $k \geq 1$ then $\phi = 0$.*

Proof. Let $\phi \in X$ and we suppose $L(\phi) = 0$. Writing ϕ as $\phi(x) = \sum_{k=1}^\infty a_k(r)\psi_k(\theta)$ and writing out $L(\phi) = 0$ gives

$$0 = \sum_{k=1}^\infty \left(-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + (\nu(N-2+\nu) + \mu_k)\frac{a_k(r)}{r^2} \right) \psi_k(\theta),$$

and hence we must have

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + (\nu(N-2+\nu)+\mu_k)\frac{a_k(r)}{r^2} = 0 \quad 0 < r < \infty,$$

which are ode's of Euler type. Define

$$\gamma_k := -(\mu_k + \nu(N-2+\nu)),$$

and hence we can re-write the above ode's as

$$a_k''(r) + \frac{(N-1)}{r}a_k(r) + \gamma_k \frac{a_k(r)}{r^2} = 0 \quad 0 < r < \infty.$$

Looking for solutions of the form $a(r) = r^\beta$ we see β needs to satisfy

$$\beta^2 + (N-2)\beta + \gamma_k = 0,$$

and from this we need to consider three cases:
Case I: $(N-2)^2 - 4\gamma_k > 0$

Case II: $(N-2)^2 - 4\gamma_k = 0$
Case III: $(N-2)^2 - 4\gamma_k < 0$.

Case I. In this case we have

$$\beta_k^+ := \frac{-(N-2)}{2} + \frac{\sqrt{(N-2+2\nu)^2 + 4\mu_k}}{2}$$

$$\beta_k^- := \frac{-(N-2)}{2} - \frac{\sqrt{(N-2+2\nu)^2 + 4\mu_k}}{2},$$

and hence $a_k(r) = C_k r^{\beta_k^+} + D_k r^{\beta_k^-}$ for some $C_k, D_k \in \mathbb{R}$ and note $\beta_k^- < \beta_k^+$. To show $C_k = D_k = 0$ it will be sufficient to show that both of β_k^+, β_k^- differ from ν. First note that if $\beta_k^+ = \nu$ then we have

$$N - 2 + 2\nu = \sqrt{(N-2+2\nu)^2 + 4\mu_k}$$

and if $p < \frac{N+2}{N-2}$ then the left hand side is negative; a contradiction. So we now assume $p > \frac{N+2}{N-2}$. Then by squaring both sides we see that $\beta_k^+ = \nu$ exactly when $\mu_k = 0$.

We now examine when $\beta_k^- = \nu$. Note we have equality when

$$-(N-2) - 2\nu = \sqrt{(N-2+2\nu)^2 + 4\mu_k}$$

and note the left hand side is negative when $\frac{N+2}{N-2} < p$ and hence we can restrict $p < \frac{N+2}{N-2}$ and in this case we see we have equality exactly when $\mu_k = 0$.

Case II. $(N-2)^2 - 4\gamma_k = 0$. In this case we

$$a_k(r) = C_k r^{\frac{-(N-2)}{2}} + D_k r^{\frac{-(N-2)}{2}} \ln(r),$$

and provided $p \neq \frac{N+2}{N-2}$ we can again show $C_k = D_k = 0$ by sending $r \to 0$ or ∞. Note here we don't need assume $\mu_k \neq 0$.

Case III. $(N-2)^2 - 4\gamma_k < 0$. In this case define

$$\omega_k := \frac{\sqrt{4\gamma_k - (N-2)^2}}{2}$$

and then the general solution given by

$$a_k(r) = C_k r^{\frac{2-N}{2}} \sin(\omega_k \ln(r)) + D_k r^{\frac{2-N}{2}} \cos(\omega_k \ln(r)),$$

for constants C_k, D_k. As in Case II we see we must have $C_k = D_k = 0$ after considering sending r to 0 and ∞; also note we are not assuming $\mu_k \neq 0$.

$\square$

We now state our main result regarding the linear operator L.

Proposition 1. *Suppose $N \geq 3$, $p \in (p_0, p_2)$ and $p \neq \frac{N+2}{N-2}$ and let w_p denote the positive classical radial solution promised by Theorem A. Suppose $\mu_k \neq 0$ for all $k \geq 1$. Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X$ such that $L(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$ and $\|\phi\|_X \leq C\|f\|_Y$.*

Note that $\mu_k \neq 0$ for all $k \geq 1$ is just the statement that w_p is a nondegenerate solutions of (3). Hence this result is saying provided w_p is a nondegenerate solution then L has good mapping properties from X to Y.

Proof. We write f and ϕ as $f(x) = \sum_{k=1}^{\infty} b_k(r)\psi_k(\theta)$ and $\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$ and hence we need to find a_k such that

$$a_k''(r) + \frac{(N-1)a_k'(r)}{r} - (\nu(N-2+\nu) + \mu_k)\, \frac{a_k(r)}{r^2} = b_k(r), \qquad 0 < r < \infty.$$

As before we need to separate the three cases:
Case I: $(N-2)^2 - 4\gamma_k > 0$
Case II: $(N-2)^2 - 4\gamma_k = 0$
Case III: $(N-2)^2 - 4\gamma_k < 0$.

Case I. We now assume we are in Case I.
Homogenous solutions. $y_1(r) = r^{\beta_k^-}$, $y_2(r) = r^{\beta_k^+}$ and the Wronskian is then $W(r) = (\beta_k^+ - \beta_k^-)r^{1-N}$.

A particular solution. A particular solution is given by $a_k(r) = y_1(r)u(r) + y_2(r)v(r)$ where

$$u'(r) = \frac{-y_2 b}{W} = \frac{-b_k(r)r^{\beta_k^+ + N-1}}{\beta_k^+ - \beta_k^-}, \qquad v'(r) = \frac{y_1 b}{W} = \frac{b_k(r)r^{\beta_k^- + N-1}}{\beta_k^+ - \beta_k^-}.$$

Put $\gamma_k := -(\mu_k + \nu(N-2+\nu))$.

$$a_k(r) = r^{\beta_k^-} \int_{T_2}^{r} \frac{y_1(t)b_k(t)}{W(t)}dt - r^{\beta_k^+} \int_{T_1}^{r} \frac{y_2(t)b_k(t)}{W(t)}dt$$

where we are free to choose T_k and then

$$a_k(r) = \frac{r^{\beta_k^-}}{\beta_k^- - \beta_k^+} \int_{T_2}^{r} \frac{b_k(t)}{t^{\beta_k^- - 1}}dt - \frac{r^{\beta_k^+}}{\beta_k^- - \beta_k^+} \int_{T_1}^{r} \frac{b_k(t)}{t^{\beta_k^+ - 1}}dt.$$

We now get estimates on the solution. Depending on the sign of $\beta_k^+ - \nu$ and $\beta_k^- - \nu$ (note its nonzero by assumption) we pick $T_i = 0$ or $T_i = \infty$. For instance lets assume $\nu - \beta_k^+ < 0$ and we then pick $T_1 = \infty$. Lets assume $|b_k(t)|t^{2-\nu} \leq 1$ and then note we have

$$r^{\beta_k^+} \int_{r}^{\infty} \frac{|b_k(t)|}{t^{\beta_k^+ - 1}}dt \leq r^{\beta_k^+} \int_{r}^{\infty} t^{\nu - \beta_k^+ - 1}dt \leq C_k r^{\nu},$$

and similarly for the other term. This shows that $|a_k(r)|r^{-\nu} \leq C_k$ for $r > 0$. This gives us a bound on the first term in the X norm on the k^{th} mode of ϕ.

Case II. $(N-2)^2 - 4\gamma_k = 0$ which we can re-write at $\mu_k(p) = \frac{-(N-2+2\nu)^2}{4}$. As above we assume $t^{2-\nu}|b_k(r)| \leq 1$ and we now write

$$a_k(r) = \ln(r) r^{\frac{2-N}{2}} \int_{T_2}^r t^{\frac{N}{2}} b_k(t)dt - r^{\frac{2-N}{2}} \int_{T_1}^r \ln(t) t^{\frac{N}{2}} b_k(t)dt.$$

First assume $1 < p < \frac{N+2}{N-2}$ and note that in this case we have $\frac{N}{2} + \nu - 1 < 0$. Take $T_1 = T_2 = \infty$, then using the above formula we can write

$$a_k(r) = r^{\frac{2-N}{2}} \int_{\infty}^r t^{\frac{N}{2}} b_k(t)(\ln(r) - \ln(t))dt.$$

Hence,

$$|a_k(r)| \leq r^{\frac{2-N}{2}} \int_r^{\infty} t^{\frac{N}{2}} |b_k(t)|(\ln(t) - \ln(r))dt.$$

Fix an $0 < \varepsilon < -(\frac{N}{2} + \nu - 1)$ then using the inequality

$$\ln(t) - \ln(r) = \ln \frac{t}{r} \leq C_\varepsilon \left(\frac{t}{r} \right)^\varepsilon \quad t < r \;\; with \;\; C_\varepsilon := \frac{1}{\varepsilon e}$$

we obtain

$$\begin{aligned} |a_k(r)| &\leq C_\varepsilon r^{\frac{2-N}{2}} \int_r^{\infty} t^{\frac{N}{2}} |b_k(t)| \left(\frac{t}{r} \right)^\varepsilon dt \\ &= C_\varepsilon r^{\frac{2-N}{2} - \varepsilon} \int_r^{\infty} t^{\frac{N}{2} + \nu - 2 + \varepsilon} |t^{2-\nu} b_k(t)| dt, \\ &\leq C_k r^{\frac{2-N}{2} - \varepsilon} \int_r^{\infty} t^{\frac{N}{2} + \nu - 2 + \varepsilon} dt \leq C_k r^\nu. \end{aligned}$$

Note that in the above we used that $\frac{N}{2} + \nu - 2 + \varepsilon < -1$.
Now consider the case $\frac{N+2}{N-2} < p < \frac{N+1}{N-3}$. Then note that in this case we have $\frac{N}{2} + \nu - 1 > 0$. Take $T_1 = T_2 = 0$, then we have

$$|a_k(r)| \leq r^{\frac{2-N}{2}} \int_0^r t^{\frac{N}{2}} |b_k(t)|(\ln(r) - \ln(t))dt.$$

Fix $0 < \varepsilon < \frac{N}{2} + \nu - 1$ and then similar as above we obtain

$$|a_k(r)| \leq C_\varepsilon r^{1+\varepsilon-\frac{N}{2}} \int_0^r t^{\frac{N}{2} + \nu - 2 - \varepsilon} |t^{2-\nu} b_k(t)| dt,$$

and since $\frac{N}{2} + \nu - 2 - \varepsilon > -1$ we get

$$|a_k(r)| \leq C_k r^\nu.$$

Case III. Lets assume we are in Case III. We use variation of parameters and hence we must first find the homogenous solutions; y_i. A computation shows $y_1(r) = r^{\frac{2-N}{2}} \sin(\omega_k \ln(r))$ and $y_2(r) = r^{\frac{2-N}{2}} \cos(\omega_k \ln(r))$ where $\omega_k := \frac{\sqrt{4\gamma_k - (N-2)^2}}{2}$, and the Wronskian $W(r) = -\omega_k r^{1-N}$. We then have a particular solution of the form $a_k(r) = y_1(r)u(r) + y_2(r)v(r)$ where

$$u'(r) = \frac{-b_k y_2}{W} = \frac{b_k(r) r^{\frac{N}{2}} \cos(\omega_k \ln(r))}{\omega_k}, \qquad v'(r) = \frac{b_k y_1}{W} = \frac{-b_k(r) r^{\frac{N}{2}} \sin(\omega_k \ln(r))}{\omega_k}.$$

Doing the computations we arrive at

$$\begin{aligned} \omega_k a_k(r) &= r^{\frac{2-N}{2}} \sin(\omega_k \ln(r)) \int_{T_1}^{r} b_k(t) t^{\frac{N}{2}} \cos(\omega_k \ln(t)) dt \\ &\quad - r^{\frac{2-N}{2}} \cos(\omega_k \ln(r)) \int_{T_2}^{r} b_k(t) t^{\frac{N}{2}} \sin(\omega_k \ln(t)) dt, \end{aligned}$$

where we can pick T_1, T_2. Take $T_1 = T_2 = \infty$ and so we have

$$\begin{aligned} \omega_k a_k(r) &= r^{\frac{2-N}{2}} \sin(\omega_k \ln(r)) \int_{\infty}^{r} b_k(t) t^{\frac{N}{2}} \cos(\omega_k \ln(t)) dt \\ &\quad - r^{\frac{2-N}{2}} \cos(\omega_k \ln(r)) \int_{\infty}^{r} b_k(t) t^{\frac{N}{2}} \sin(\omega_k \ln(t)) dt, \end{aligned}$$

and we suppose $|b_k(t)| t^{2-\alpha} \le 1$; and note the integrals are well defined for any $r > 0$ since the bound on b_k shows the integrands are integrable on (r, ∞). Using the bounds on b_k we also see that there is some $C_k > 0$ such that $r^{-\nu} |a_k(r)| \le C_k$ for $r > 0$. In Case I, II and III we can use the ode for a_k directly along with the zero order bounds on a_k to see that $\|a_k\|_{C^{2,\alpha}_{\nu}} \le C_k \|b_k\|_{C^{0,\alpha}_{\nu-2}}$.

So far we have shown for each $k \ge 1$ and $f_k(x) = b_k(r)\psi_k(\theta)$ with $f_k \in Y$ there is some $\phi_k \in X$, $\phi_k(x) = a_k(r)\psi_k(\theta)$ which satisfies $L(\phi_k) = f_k$ in Ω with $\phi_k = 0$ on $\partial\Omega$. Moreover there is some $C_k > 0$ (independent of f_k, a_k) such that $\|\phi_k\|_X \le C_k \|f_k\|_Y$.

One can ague that for each $m \ge 1$ there is some $D_m > 0$ such that

$$\sum_{k=1}^{m} \|b_k \psi_k\|_Y \le D_m \| \sum_{k=1}^{m} b_k \psi_k \|_Y$$

independent of b_k and hence using this we see for any

$$f_m(x) := \sum_{k=1}^{m} b_k(r) \psi_k(\theta)$$

there is some $\phi_m(x) = \sum_{k=1}^{m} a_k(r)\psi_k(\theta)$ such that $L(\phi_m) = f_m$ on Ω with $\phi_m = 0$ on $\partial\Omega$ and $\|\phi_m\|_X \le D_m \|f_m\|_Y$. We now show we can take D_m independent of m in $\|\phi_m\|_X \le D_m \|f_m\|_Y$.

Suppose not, then there is some f_m (as above) and ϕ_m with $\|\phi_m\|_X = 1$ and $\|f_m\|_Y \to 0$. We begin by showing that the zero order term of the norm of ϕ_m must go to zero. Let $s_m > 0$ be arbitrary and set

$$\varepsilon_m := s_m^{-\nu} \sup_{A_{s_m}} |\phi_m|,$$

and define $\psi_m(x) := s_m^{-\nu}\phi_m(s_m x)$. Note that ψ_m satisfies

$$-\Delta\psi_m(x) - \frac{pw(\theta)^{p-1}\psi_m(x)}{|x|^2} = g_m(x) \qquad \text{in } \Omega \tag{18}$$

with $\psi_m = 0$ on Ω where $g_m(x) := s_m^{2-\nu} f_m(s_m x)$. A computation shows for each k we have

$$\frac{|g_m(x) - g_m(y)|}{|x-y|^\alpha} \le k^{2-\nu+\alpha}\|f_m\|_Y \quad \forall x, y \in E_k := \{x \in \Omega : \frac{1}{k} < |x| < k\},$$

and $\sup_{E_k} |g_m(x)| \le k^{2-\nu}\|f_m\|_Y$. This implies that $g_m \to 0$ in $C^{0,\alpha}(\overline{E_k})$ for each $k \ge 1$. Additionally using the bound $\|\phi_m\|_X \le 1$ we have $|\psi_m(x)| \le \frac{2^{-\nu}}{|x|^{-\nu}}$ on Ω. Also, it is not hard to obtain that $\|\nabla\psi_m\|_{L^\infty(E_k)} \le k^{1-\nu}$ and $\|D^2\psi_m\|_{L^\infty(E_k)} \le k^{2-\nu}$, and also $k^{2-\nu+\alpha}$ as the uniform upper bound for the Hölder norm of $D^2\psi_m$ on E_k. Using a diagonal argument we can then find some ψ such that $\psi_m \to \psi$ in $C^{2,\beta}(\overline{E_k})$ for $\beta < \alpha$ and all $k \ge 2$ and ψ satisfies

$$-\Delta\psi(x) - \frac{pw(\theta)^{p-1}\psi(x)}{|x|^2} = 0 \quad \text{in } \Omega,$$

with $\psi = 0$ on $\partial\Omega\backslash\{0\}$. Note additionally that $|\psi(x)| \le \frac{2^{-\nu}}{|x|^{-\nu}}$. Using this zero order bound along with a scaling argument one can show that $\psi \in X$ and hence by Lemma 1 we have $\psi = 0$ and so $\psi_m \to 0$ in $C^{2,\beta}(\overline{E_k})$ for each $k \ge 2$. But note we have

$$s_m^{-\nu} \sup_{A_{s_m}} |\phi_m(x)| \le \sup_{\frac{1}{2}|<|x|<2} s_m^{-\nu}|\phi_m(s_m x)| = \sup_{x\in E_2} |\psi_m(x)| \to 0,$$

and hence we have the zero order portion of the norm goes to zero.

We now show the other portions also go to zero. Let $s_m > 0$ be arbitrary and we re-write (18) as

$$-\Delta\psi_m(x) = g_m(x) + \frac{pw(\theta)^{p-1}\psi_m(x)}{|x|^2} \qquad \text{in } E_4,$$

with $\psi_m = 0$ on the lateral boundary of E_4. Note that from the above estimates we have that the right hand side of this equation converges to zero in $C^{0,\alpha}(\overline{E_4})$ and hence by elliptic regularity theory we have $\psi_m \to 0$ in $C^{2,\alpha}(\overline{E_2})$. After scaling back and the fact that $s_m > 0$ is arbitrary we see that $\phi_m \to 0$ in X, which contradicts the bound $\|\phi_m\|_X = 1$.

□

3 Nondegeneracy of w_p

3.1 Nondegeneracy of w_p on (p_1, p_2) when S is the geodesic ball

In the case of S a geodesic ball, recall from Theorem A [38] (see Section 1.2) we have w_p is a positive nondegenerate solution of (3) for $p \in (p_0, p_1)$. In the current section we will apply bifurcation theory to partially extend this result to the full interval (p_0, p_2). We follow the setting from [4], which follows closely the book of [5] and also the paper [12].

Let $\mathcal{X}, \mathcal{Y}$ denote Banach spaces, $\mathcal{U} \subset \mathbb{R} \times \mathcal{X}$ an open set containing $(0,0)$ in its closure and $F : \mathcal{U} \to \mathcal{Y}$ an $\mathbb{R}$ analytic function. We define

$$\mathcal{S} := \{(\lambda, x) \in \mathcal{U} : F(\lambda, x) = 0\}, \quad \text{and}$$

$$\mathcal{N} := \{(\lambda, x) \in \mathcal{S} : Ker(\partial_x F(\lambda, x) = \{0\}\},$$

and we define a *distinguished arc* to be a maximal connected subset of $\mathcal{N}$. We now define some conditions:

(G1) Bounded closed subsets of $\mathcal{S}$ are compact in $\mathbb{R} \times \mathcal{X}$.
(G2) $\partial_x F(\lambda, x)$ is a Fredholm operator of index zero for all $(\lambda, x) \in \mathcal{S}$.
(G3) There exists an analytic function $(\lambda, u) : (0, \varepsilon) \to \mathcal{S}$ such that $\partial_x F(\lambda(s), u(s))$ is invertible for all $s \in (0, \varepsilon)$ and $\lim_{s \to 0^+}(\lambda(s), u(s)) = (0,0)$.

Define $\mathcal{A}^+ := \{\lambda(s), u(s) : s \in (0, \varepsilon)\}$.

We now state Theorem 1.13 from [4].

Theorem 1.13. *[4] Suppose (G1)-(G3) hold. Then,* (λ, u) *can be extended as a continuous map (still called)* $(\lambda, u) : (0, \infty) \to \mathcal{S}$ *with the following properties:*

1. *Define* $\mathcal{A} := \{(\lambda(s), u(s)) : s > 0\}$*. Then* $\mathcal{N} \cap \mathcal{A}$ *is an at most countable union of distinct distinguished arcs* $\cup_{i=0}^{n} \mathcal{A}_i$, $n \leq \infty$.
2. $\mathcal{A}^+ \subset \mathcal{A}_0$.
3. $\{s > 0 : Ker(\partial_x F(\lambda(s), u(s)) \neq \{0\}$ *is a discrete set.*
4. *At each of its points* $\mathcal{A}$ *has a local analytic re-parameterization (see [4] for details).*
5. *One of the following occurs.*
   (a) $\|(\lambda(s), u(s))\|_{\mathbb{R} \times \mathcal{X}} \to \infty$ *as* $s \to \infty$.
   (b) *the sequence* $\{(\lambda(s), u(s))\}$ *approaches the boundary of* $\mathcal{U}$ *as* $s \to \infty$.

(c) $\mathcal{A}$ *is the closed loop:*

$$\mathcal{A} = \{(\lambda(s), u(s)) : 0 \leq s \leq T, (\lambda(T), u(T)) = (0,0) \text{ for some } T > 0\}.$$

In this case, choosing the smallest such $T > 0$ *we have* $(\lambda(s+T), u(s+T)) = (\lambda(s), u(s))$ *for all* $s \geq 0$.

6. *Suppose* $\partial_x F(\lambda(s_1), u(s_1))$ *is invertible for some* $s_1 > 0$. *If for some* $s_2 \neq s_1$, *we have* $(\lambda(s_1), u(s_1)) = (\lambda(s_1), u(s_2))$ *then 5 (c) occurs and* $|s_2 - s_1|$ *is an integer multiple of* T. *In particular the map* $s \mapsto (\lambda(s), u(s))$ *is injective on* $[0, T)$.

We will now apply this theorem to (3); which we re-write for the sake of the reader (and we change notation to u to agree with the above stated theorem)

$$\begin{cases} -\Delta_\theta u &= \lambda_p u + u^p \qquad \text{in } S, \\ \quad u &= 0 \qquad\qquad\quad \text{on } \partial S. \end{cases}$$

Let $\phi_1 > 0$ denote the first eigenfunction of $-\Delta_\theta$ in $H_0^1(S)$, which is L^∞ normalized. We define

$$C_{\phi_1} := \left\{ u \in C_0(S) : \|u\|_{C_{\phi_1}} := \sup_\Omega \frac{|u|}{\phi_1} < \infty \right\},$$

and we set

$$C_{\phi_1}^+ := \left\{ u \in C_{\phi_1} : \inf_\Omega \frac{u}{\phi_1} > 0 \right\},$$

which is open in C_{ϕ_1}. We now set $\mathcal{X} = \mathcal{Y} := C_{\phi_1}, \mathcal{U} := (p_0, p_2) \times C_{\phi_1}^+$ and define the mapping $F : \mathcal{U} \to Y$ by

$$F(p, u) := u + \lambda_p (\Delta_\theta)^{-1} u + (\Delta_\theta)^{-1} u^p.$$

(G1') Since we are working on the finite interval (p_0, p_2) we can adjust (G1) to read for all $\delta > 0$ (small) that $\mathcal{S}_\delta := \{(p, u) \in \mathcal{U} : F(p, u) = 0, p \in [p_0 + \delta, p_2 - \delta]\}$ is compact in $(p_0, p_2) \times \mathcal{X}$.

Condition (G1') We begin by checking the condition (G1'). We suppose the result is false and so there is some (p_m, u_m) such that $p_m \to \hat{p} \in (p_0, p_2)$ and u_m does not converge in $\mathcal{X}$. First note that if $\|u_m\|_{L^\infty}$ is unbounded then a blow up argument allows one to obtain the needed contradiction; see Lemma 2. So from this we see that u_m is uniformly bounded in L^∞. If $\|u_m\|_{L^\infty} \to 0$ we can renormalize to find a positive solution v of $-\Delta_\theta v = \lambda_{\hat{p}} v$ in S with $v = 0$ on ∂S; which is a contradiction since $\lambda_{\hat{p}}$ is not equal the first eigenfunction of $-\Delta_\theta$ in $H_0^1(S)$. So we have shown there are positive constants C_i with $C_1 \leq \|u_m\|_{L^\infty} \leq C_2$. By elliptic regularity we can pass to a subsequence and find some $\varepsilon_0 > 0$ and $u \in C^{2,\varepsilon_0}(S)$ such that $u_m \to u$ in $C^{2,\varepsilon_0}(S)$. Note that $u \in C_{\phi_1}$ after considering the fact that the gradient is bounded. This convergence is sufficient to show that $u_m \to u$ in C_{ϕ_1}; a contradiction. So we have shown condition (G1') holds.

Condition (G2) Condition (G2) holds from standard abstract elliptic theory.

Condition (G3) We first need to prove the existence of a local branch of solutions and to do this we apply the Crandall-Rabinowitz Theorem, see Theorem 1.7 in [10] and for the readers convenience we restate it here.

Theorem 1.7. *[10] Let X, Y be Banach spaces, V a neighborhood of 0 in X and*

$$F : (-1,1) \times V \to Y$$

have the properties
(a) $F(t,0) = 0$ for $|t| < 1$,
(b) the partial derivatives F_t, F_x and F_{tx} exists and are continuous,
(c) $N(F_x(0,0))$ and $\frac{Y}{R(F_x(0,0))}$ are one-dimensional.
(d) $F_{tx}(0,0)x_0 \notin R(F_x(0,0))$, where

$$N(F_x(0,0)) = span\{x_0\}.$$

If Z is any complement of $N(F_x,(0,0))$ in X, then there is a neighborhood U of $(0,0)$ in $\mathbb{R} \times X$, an interval $(-a,a)$, and continuous functions $\varphi : (-a,a) \to \mathbb{R}$, $\psi : (-a,a) \to Z$ such that $\varphi(0) = 0$, $\psi(0) = 0$ and

$$F^{-1}\{0\} \cap U = \{(\varphi(s), sx_0 + s\psi(s)) : \ |s| < a\} \cup \{(t,0) : \ (t,0) \in U\}.$$

If F_{xx} is also continuous, the functions φ and ψ are once continuously differentiable.

To apply the above theorem for our equation, let $X, Y = C_{\phi_1}$ and consider the operator $F : (1, p_2) \times X \to Y$ as

$$F(p,u) := u - \lambda_p(-\Delta_\theta)^{-1}u - (-\Delta_\theta)^{-1}|u|^p.$$

We have $F(p,0) = 0 \in Y$ *for all* $p \in (1,p_2)$. Also, at the point p_0 we have $\lambda_{p_0} = \lambda_1(S)$ and

$$L\xi := F_u(p_0,0)\xi = \xi - \lambda_1(-\Delta_\theta)^{-1}\xi,$$

which is a Fredholm operator of index zero and $ker(L)$ is one-dimensional, indeed we have $ker(L) = span\{\phi_1\}$ where, as before, $\phi_1 > 0$ is the first eigenfunction of $-\Delta_\theta$ in $H_0^1(S)$ (which is L^∞ normalized). Additionally, we have that codim $Rang(L) = 1$ by the Fredholm alternative. Now we check the transversality condition:

$$\partial^2_{p,u}F(p_0,0)\phi_1 \notin range(L). \tag{19}$$

We have

$$\begin{aligned} \partial^2_{p,u}F(p_0,0)\phi_1 &= \lim_{t\to 0} \frac{\partial_u F(p_0+t,0)\phi_1 - \partial_u F(p_0,0)\phi_1}{t} \\ &= \lim_{t\to 0} \frac{\lambda_{p_0} - \lambda_{p_0+t}}{t}(-\Delta_\theta)^{-1}\phi_1 \\ &= -\left[\frac{\lambda_1}{p_0-1} + \frac{4}{(p_0-1)^3}\right]\frac{\phi_1}{\lambda_1} := \frac{\beta}{\lambda_1}\phi_1, \end{aligned}$$

so if (19) does not hold then there should be a $\xi \in X$ such that $\xi - \lambda_1(-\Delta_\theta)^{-1}\xi = \frac{\beta}{\lambda_1}\phi_1$ or

$$-\Delta\xi = \lambda_1\xi - \beta, \ in \ S \ \ \xi = 0 \ on \ \partial S,$$

which is impossible because $\beta \neq 0$. Thus by the above Theorem 1.7, there exists a continuous function

$$(p(s), u(s)) : (-a, a) \to (1, p_1) \times X$$

such that $F(p(s), u(s)) = 0$ for all $s \in (-a, a)$, $p(0) = p_0$, and $u(s) = s\phi_1 + s\psi(s)$, where ψ is continuous on $(-a, a)$, $\psi(0) = 0$ and ψ is orthogonal to ϕ_1. Now note by the continuity of ψ at $s = 0$ we have

$$\sup_{x \in S} \frac{|\psi(s)|}{\phi_1}(x) = \|\psi(s)\|_{C_{\phi_1}} = \|\psi(s) - \psi(0)\|_{C_{\phi_1}} < \frac{1}{2},$$

for s sufficiently small, gives that $u(s) \geq \frac{s}{2}\phi_1$, for $s > 0$ small. In particular, $u(s) \in C^+_{\phi_1}$ for $s > 0$ sufficiently small. Note also that for $0 < s$ small we must have $p(s) > p_0$; otherwise we would not have a positive solution.

We need to now show that this local solution is analytic in the parameter s. We now return to the prior setting of $F : \mathcal{U} \to C_{\phi_1}$ and we can apply a similar argument to [12] to see that the solution is analytic in the parameter s on an interval of the form $s \in (0, \varepsilon)$. To see the nondegeneracy condition we suppose that $(p(s_m), u(s_m)) = (p_m, u_m)$ is a degenerate solution with $s_m \to 0$ and we let ϕ_m denote L^∞ normalized solutions of the linearized equation. Hence we have

$$-\Delta_\theta \phi_m = \lambda_{p_m}\phi_m + p_m u_m^{p_m - 1}\phi_m \quad \text{in } S, \qquad \phi_m = 0 \quad \text{on } \partial S,$$

and we have $\|u_m\|_{L^\infty} \to 0$. Since $p_m \to p_0$ we see $p_m - 1$ is bounded away from zero and hence we can pass to the limit in the equation to see that there is some ϕ_∞ which is L^∞ normalized such that $\phi_m \to \phi_\infty$ in $C^{2,\varepsilon_0}(S)$ (after passing to a subsequence) and which satisfies

$$-\Delta_\theta \phi_\infty = \lambda_1(S)\phi_\infty \quad \text{in } S, \qquad \phi_\infty = 0 \quad \text{on } \partial S,$$

and hence we must have ϕ_∞ to be either ϕ_1 or $-\phi_1$. We can assume without loss of generality that $\phi_\infty = \phi_1$ and hence we have $\phi_m \to \phi_1$ in $C^{2,\varepsilon_0}(S)$. Multiplying the equation for u_m by ϕ_m and integrating and then multiplying the equation for ϕ_m by u_m and integrating (and equating) we arrive at

$$\int_S u_m^{p_m}\phi_m d\theta = 0,$$

for all m; and hence ϕ_m must be sign changing. Using the convergence of ϕ_m we see $\phi_m > 0$ in S for large enough m and hence we have a contradiction. From this we see that by taking $\varepsilon > 0$ smaller if necessary we have that $(u(s), p(s))$ is a nondegenerate solution for all $s \in (0, \varepsilon)$. This proves that condition (G3) holds. For later reference we now define $\mathcal{A}^+ := \{(p(s), u(s)) : s \in (0, \varepsilon)\}$.

We now apply Theorem 1.13 [4] to obtain an extension of the map $s \mapsto (p(s), u(s))$ to all of $(0, \infty)$. In particular part 3 gives us that $\{s > 0 : Ker(\partial_u F(p(s), u(s))) \neq \{0\}\}$ is a discrete set. In particular if $\{p(s) : s > 0\} = (p_0, p_2)$ then we are done. To show this is the case we need to consider the 3 possible cases from alternative offered in part 5 of the theorem.

(a) We first suppose $\|(p(s), u(s))\|_{\mathbb{R} \times C_{\phi_1}} \to \infty$. By passing to a subsequence we can assume there is some $s_m \to \infty$ such that $p(s_m) \to \hat{p} \in [p_0, p_2]$. If $\hat{p} = p_2$ then we are done so we now assume $\hat{p} \in [p_0, p_2)$ and we let $t_m := \|u_m\|_{L^\infty}$. If $t_m \to \infty$ we can apply a blow up argument to obtain a contradiction; see Lemma 2. We now suppose t_m is bounded above. Then by elliptic regularity we see that u_m is bounded in $C^{1,\delta}(\overline{S})$ for some $\delta > 0$ and this is sufficient to see that u_m is bounded in C_{ϕ_1} and hence we obtain the needed contradiction. So we have shown that either case (a) happens and $\{p(s) : s > 0\} = (p_0, p_2)$ (hence we are done) or case (a) cannot happen.

(b) We now consider the possibility of case (b) happening. Lets suppose after passing to a subsequence we have $p(s_m) \to p_2$; then as before we are done. So lets assume $p(s) \to p_0$. Then there is some s_0 (large) such that for all $s \geq s_0$ we have $p(s) \in (p_0, p(\varepsilon))$ and by uniqueness of solution for $p \in (p_0, p_1)$ Theorem A [38] we see there is some $\tau_s \in (0, \varepsilon)$ such that $(p(s), u(s)) = (p(\tau_s), u(\tau_s))$. Therefore, we obtain that the distinguished arc corresponding to all large s must coincide with $\mathcal{A}_0$. This gives a contradiction, as distinguished arcs are distinct from Theorem 1.13 part 1.

So we can now suppose $\hat{p} \in (p_0, p_2)$ and we set $t_m := \|u_m\|_{L^\infty}$. As before if $t_m \to \infty$ we can obtain a contradiction. If $t_m \to 0$ we can re-normalize u_m by $v_m(x) := \frac{u_m(x)}{t_m}$ and see that $v_m > 0$ converges to some $v > 0$ with $\|v\|_{L^\infty} = 1$ and $-\Delta_\theta v = \lambda_{\hat{p}} v$ in S with $v = 0$ on ∂S; but this gives us a contradiction since $\lambda_{\hat{p}} \neq \lambda_1(S)$. So now we can assume t_m is bounded and bounded away from zero. From this and elliptic regularity we can find a some $0 < u \in C^{2,\varepsilon_0}(S)$ which satisfies

$$-\Delta_\theta u = \lambda_{\hat{p}} u + u^{\hat{p}} \quad \text{in } S, \qquad u = 0 \quad \text{on } \partial S,$$

and by Hopf's lemma we have $\inf_{\partial S} |\nabla u| > 0$. From this we can conclude that $u \in C^+_{\phi_1}$, and using the above stated convergence we see that $u_m \to u$ in C_{ϕ_1}; and hence we cannot have u_m converge to the boundary of $C^+_{\phi_1}$. Hence we have shown if we are in case (b) we must have $p(s) \to p_2$ and we are done.

(c) We now suppose we are in the closed loop case. Then for arbitrary large s with $p(s) \in (p_0, p(\varepsilon))$ we have the existence of $\tau_s \in (0, \varepsilon)$ such that $(p(s), u(s)) = (p(\tau_s), u(\tau_s))$. This shows that for these large values of s we must have $s \in \mathcal{A}_0$; which again is a contradiction by the same argument as in part (c).

So from the above we have shown the following result.

Proposition 2. *Suppose S and N are as in Theorem 1. Then there is a sequence of increasing q_k (possibly empty or finite) with $p_1 \leq q_1 < q_2 < \ldots$ with $q_k < p_2$ (with p_2 being* *the only possible limit point) such that for all $p \in (p_0, p_2) \backslash \{q_k : k \geq 1\}$ one has that w_p is a nondegenerate solution of (3).*

3.2 Nondegeneracy of w on general domains S

In this section we extend the nondegeneracy result to general domains S, but of course now the set of $\{q_k : k \geq 1\}$ where w_p is degenerate is contained in (p_0, p_2) with p_2 being the only possible limit point. To obtain this result we will again apply Theorem 1.13 [4], but first we will require a couple of results. We begin by proving classical solutions of (3) are uniformly bounded; which is really just the standard subcritical result.

Lemma 2. *(Subcritical solutions are uniformly bounded) Suppose $N \geq 3$ and $S \subset S^{N-1}$ with smooth nonempty boundary. Suppose $p_m \in [p_0, p_2)$ with $\sup_m p_m < p_2$. Then there is some $C > 0$ such that for $u_m > 0$ a classical solution of (3) we have $\|u_m\|_{L^\infty} \leq C$ for all m.*

Proof. Using the stereographic projection (see (13) we see that there is some domain $D \subset \mathbb{R}^{N-1}$ (bounded with smooth boundary) and positive solution $v_m(x)$ related to u_m via

$$v_m(x) = \frac{u_m(P^{-1}x)}{(1 + |x|^2)^{\frac{N-3}{2}}}, \qquad x \in D,$$

and v_m satisfies

$$\begin{cases} \Delta v_m + \frac{(N-1)(N-3)+4\lambda_{p_m}}{(1+|x|^2)^2} v_m + 4(1 + |x|^2)^{\frac{p(N-3)-(N+1)}{2}} v_m^{p_m} = 0 & \text{in } D, \\ v_m = 0 & \text{on } \partial D. \end{cases} \tag{20}$$

We can now use a standard blow up argument to obtain a contradiction and hence v_m must be uniformly bounded. □

The following proposition 3 is heavily inspired by [11], where they prove some uniqueness and nondegeneracy results for $-\Delta u = u^p$ in a Euclidean domain with zero boundary condition; for p close to 1. This result is also contained in [12]. We mention that the result we need is much easier than the above mentioned problem since our exponent doesn't get close to 1.

Proposition 3. *Let $S \subset S^{N-1}$ with smooth boundary with $N \geq 3$. Then there is some $\varepsilon > 0$ (small) such that for all $p \in (p_0, p_0 + \varepsilon)$ there is a unique positive solution of (3). Moreover the solution is nondegenerate.*

Proof. To agree with the notation from Section 3.1 we will change the variable from w to u. We first prove uniqueness. Suppose $p_m \searrow p_0$ and u_m, v_m are two positive distinct solutions of (3). Set $\zeta_m := \frac{u_m - v_m}{t_m}$ where $t_m > 0$ chosen such that $\|\zeta_m\|_{L^\infty} = 1$. We first show that u_m (and also v_m) must satisfy $\|u_m\|_{L^\infty} \to 0$. Set $T_m := \sup_S u_m$ and by Lemma 2 we know T_m is bounded. Suppose $T_m \to T \in (0, \infty)$; then we can pass to a limit in the equation to find a positive solution of (3) with $p = p_0$; but this is a contradiction if we recall that $\lambda_{p_0} = \lambda_1(S)$. Multiplying the equation for u_m by v_m and vice versa we obtain

$$0 = \int_S u_m v_m \left(u_m^{p_m} - v_m^{p_m}\right) d\theta,$$

and from this we see that ζ_m must change sign. A computation shows that

$$\begin{cases} -\Delta_\theta \zeta_m &= \lambda_{p_m} \zeta_m + C_m(\theta)\zeta_m \qquad \text{in } S, \\ \qquad \zeta_m &= 0 \qquad\qquad\qquad\qquad \text{on } \partial S, \end{cases} \tag{21}$$

where $C_m(\theta) := \frac{u_m^{p_m} - v_m^{p_m}}{u_m - v_m}$. Since p_m bounded away from one and $\|u_m\|_{L^\infty}, \|v_m\|_{L^\infty} \to 0$ we see that $\sup_S |C_m| \to 0$. By passing to a subsequence we can pass to the limit in (21) to see there is some $\|\zeta\|_{L^\infty} = 1$ such that $\zeta_m \to \zeta$ in $C^1(\overline{S})$ and ζ satisfies $-\Delta_\theta \zeta = \lambda_{p_0}\zeta$ in S with $\zeta = 0$ on ∂S. Recalling $\phi_1 > 0$ is L^∞ normalized first eigenfunction we see that we must have $\zeta = +\phi_1$ or $\zeta = -\phi_1$. Without loss of generality we take $\zeta = \phi_1$; but this contradicts fact that ζ_m is sign changing and converges to ϕ_1 in C^1. So we have shown there is a unique solution of (3) for $p > p_0$ but close. We now show the solution is nondegenerate.

So we let $u_m > 0$ denote a solution of above with $p_m \searrow p_0$ and we assume the solution is degenerate and we let ϕ_m denote an L^∞ normalized solution of the linearized equation, ie. $-\Delta_\theta \phi_m = \lambda_{p_m}\phi_m + p_m u_m^{p_m - 1}\phi_m$ in S with $\phi_m = 0$ on ∂S. Multiplying this equation by u_m and integrating; and multiplying the equation for u_m by ϕ_m and integrating we arrive at

$$0 = \int_S u_m^{p_m} \phi_m d\theta,$$

and so we see that ϕ_m must be sign changing. Now again noting that since $p_m - 1$ is bounded away from zero we have $\sup_S u_m^{p_m - 1} \to 0$. Arguing as above we can show that ϕ_m must converge in C^1 to either ϕ_1 or $-\phi_1$; but again we get a contradiction to ϕ_m being sign changing. $\square$

4 The fixed point argument

Recall we have defined $J_t(\phi) = \psi$ where ψ satisfies

$$\begin{aligned} L(\psi) &= (v_0 + \phi)^p - v_0^p - p v_0^{p-1}\phi + E_t(v_0) + E_t(\phi) \\ &=: K(\phi) + +E_t(v_0) + E_t(\phi) \qquad \text{in } \Omega \end{aligned}$$

with $\psi = 0$ on $\partial\Omega\backslash\{0\}$. To obtain a solution ϕ of (9) we will show that J_t is a contraction on B_{ε_0} where B_{ε_0} is the closed ball of radius ε_0 centered at the origin in X.
First we show that for ε_0 sufficiently small

$$\|K(\phi_2) - K(\phi_1)\|_{C^{0,\alpha}_{\nu-2}} \le C(\varepsilon_0)\|\phi_2 - \phi_1\|_{C^{2,\alpha}_\nu}, \quad where \;\; C(\varepsilon_0) \to 0 \; as \; \varepsilon_0 \to 0 \tag{22}$$

for all $\phi_2, \phi_1 \in B_{\varepsilon_0}$. To this end, we first prove that for ε_0 sufficiently small and $\phi \in B_{\varepsilon_0}$ we have

$$|\phi(x)| \le C\varepsilon_0 v_0(x), \quad \text{for} \quad x \in \Omega.$$

To see the above note that $\|\phi\|_{C^{2,\alpha}_\nu} < \varepsilon_0$ implies $|\phi(x)| < \varepsilon_0 |x|^\nu$, for $x \in \Omega \setminus \{0\}$. Now set $\phi(x) = \phi(r,\theta) = \varepsilon(r,\theta) r^\nu$ with $|\varepsilon(r,\theta)| < \varepsilon_0$, so to get $|\phi(x)| \le C\varepsilon_0 v_0(x)$ it suffices to show

$$|\varepsilon(r,\theta)| < C\varepsilon_0 w(\theta) \quad \text{for all} \quad r \in (0,\infty). \tag{23}$$

Using the inequality $\frac{|\nabla_\theta \phi|}{r} \le |\nabla_x \phi|$ in the second term in the definition of $\|\phi\|_{C^{2,\alpha}_\nu}$ we get

$$\sup_{s>0} s^{-\nu} \left(s \sup_{A_s} \left[r^{-\nu-1} |\nabla_\theta \varepsilon(r,\theta)| \right] \right) \le \varepsilon_0,$$

and from this we roughly get

$$\sup_{s>0} \sup_{A_s} |\nabla_\theta \varepsilon(r,\theta)| \le C\varepsilon_0.$$

Then using $\varepsilon(r,\theta) = 0$ on ∂S and the mean value theorem, we get

$$|\varepsilon(r,\theta)| \le C\varepsilon_0 dist(\theta, \partial S) \quad \text{for all} \quad r \in (0,\infty).$$

Hence to prove (23) we need to show

$$C_0 dist(\theta, \partial S) \le w(\theta), \quad \theta \in S. \tag{24}$$

Recall that $w(\theta) > 0$ satisfies the equation $-\Delta_\theta w - \nu(N-2+\nu) w = w^p \ge 0$ in S, then by a refinement of Hopf's lemma (Lemma 1, chapter 9 of [22]) we get $\frac{\partial w}{\partial \nu} < 0$ on ∂S that by the compactness of ∂S this easily gives $w(\theta) > C_0 dist(\theta, \partial S)$ for $\theta \in S$. Hence we have (24) for ε_0 sufficiently small.

Now since $|\frac{\phi}{v_0}| \le C\varepsilon_0 < 1$ for $\phi \in B_{\varepsilon_0}$ we can show that, using the Taylor expansion, we have for $x \in \Omega$, $\phi_2, \phi_1 \in B_{\varepsilon_0}$ and $p > 1$

$$|K(\phi_2) - K(\phi_1)|(x) \le C v_0^{p-2}(x)(|\phi_1(x)| + |\phi_2(x)|)(|\phi_2(x) - \phi_1(x)|). \tag{25}$$

To see (25), first note we have

$$\begin{aligned} K(\phi_2) - K(\phi_1) &= (v_0 + \phi_2)^p - (v_0 + \phi_1)^p - p v_0^{p-1}(\phi_2 - \phi_1) \\ &= v_0^p \Big[(1+a)^p - (1+b)^p - p(a-b) \Big], \quad \text{where} \quad a := \frac{\phi_2}{v_0} \quad \textit{and} \quad b := \frac{\phi_1}{v_0}. \end{aligned}$$

Note when $|t| < 1$ we have by Taylor expansion (or binomial expansion), for $p > 1$

$$(1+t)^p = 1 + pt + \sum_2^\infty \gamma_k t^k,$$

where γ_k are the binomial coefficients, and where this series converges absolutely provided $|t| < 1$. Hence,

$$K(\phi_2) - K(\phi_1) = v_0^p \sum_2^\infty \gamma_k (a^k - b^k).$$

Now note by the mean value theorem, for $k \geq 2$ we have

$$|a^k - b^k| = |(a^2)^{\frac{k}{2}} - (b^2)^{\frac{k}{2}}| \leq \frac{k}{2}|a^2 - b^2| \; z^{\frac{k}{2}-1}$$

for some z between a^2 and b^2, and recall we have $|a|, |b| \leq C\varepsilon_0 < 1$, hence $z < \varepsilon_1 < 1$. Therefore, applying the above estimates we get (Note $\sum_2^\infty k\gamma_k z^{\frac{k}{2}-1}$ is bounded by a C independent of z for $0 < z < \varepsilon_1$)

$$|K(\phi_2) - K(\phi_1)| \leq v_0^p \sum_2^\infty \gamma_k \frac{k}{2} |a^2 - b^2| \; z^{\frac{k}{2}-1} \leq C v_0^p |a^2 - b^2|$$

$$= C v_0^p |(\frac{\phi_2}{v_0})^2 - (\frac{\phi_1}{v_0})^2| \leq C v_0^{p-2}(|\phi_2| + |\phi_1|)|\phi_2 - \phi_1|,$$

which proves (25).
Now from (25) we have

$$|K(\phi_2) - K(\phi_1)|(x) \leq C v_0^{p-1}(x)(\frac{|\phi_1(x)|}{v_0} + \frac{|\phi_2(x)|}{v_0})(|\phi_2(x) - \phi_1(x)|) \leq C\varepsilon_0 \frac{w^{p-1}(\theta)}{|x|^2}(|\phi_2(x) - \phi_1(x)|)$$

$$\leq \frac{C\varepsilon_0}{|x|^2}(|\phi_2(x) - \phi_1(x)|)$$

Thus for $x \in A_s$ we get

$$s^{2-\nu}|K(\phi_2) - K(\phi_1)|(x) \leq C\varepsilon_0 s^{-\nu}(|\phi_2(x) - \phi_1(x)|),$$

that gives

$$\sup_{s>0} s^{2-\nu} \sup_{A_s} |K(\phi_2) - K(\phi_1)|(x) \leq C\varepsilon_0 \|\phi_2 - \phi_1\|_{C_\nu^{2,\alpha}}. \tag{26}$$

To estimate the Holder norm of $K(\phi_2) - K(\phi_1)$, first note that using the inequality

$$\sup_{s>0} s^{2-\nu+\alpha} \sup_{x,y \in A_s} \frac{|f(x) - f(y)|}{|x - y|^\alpha} \leq 4 \sup_{s>0} s^{3-\nu} \sup_{A_s} |\nabla f|,$$

it suffices to estimate $s^{3-\nu}|\nabla K(\phi_2) - \nabla K(\phi_1)|$ in A_s. We have

$$\nabla K(\phi) = p\Big((v_0 + \phi)^{p-1}) - v_0^{p-1} - (p-1)v_0^{p-2}\phi\Big)\nabla v_0 + p\Big((v_0 + \phi)^{p-1}) - v_0^{p-1}\Big)\nabla\phi.$$

Again using Taylor expansion in the expression above and similar as before one can show that

$$\sup_{s>0} s^{3-\nu} \sup_{A_s} |\nabla K(\phi_2) - \nabla K(\phi_1)| \le C(\varepsilon_0)\|\phi_2 - \phi_1\|_{C^{2,\alpha}_\nu}, \quad where \;\; C(\varepsilon_0) \to 0 \; as \; \varepsilon_0 \to 0. \tag{27}$$

Now (26) and (27) imply (22).
Also by the definition of E_t and assumptions in (6) its easy to get

$$\|E_t(\phi_2 - \phi_1)\|_{C^{0,\alpha}_{\nu-2}} \le Ct\|\phi_2 - \phi_1\|_{C^{2,\alpha}_\nu} \;\; and \;\; \|E_t(v_0)\|_{C^{0,\alpha}_{\nu-2}} \le Ct. \tag{28}$$

Now note that taking $\phi_1 = 0$ in (22) and (28) we get

$$\|K(\phi)\|_{C^{0,\alpha}_{\nu-2}} \le C(\varepsilon_0)\|\phi\|_{C^{2,\alpha}_\nu} \;\; and \;\; \|E_t(\phi)\|_{C^{0,\alpha}_{\nu-2}} \le Ct\|\phi\|_{C^{2,\alpha}_\nu}, \quad \phi \in B_{\varepsilon_0}. \tag{29}$$

Now by the definition of J_t, the continuity of the right inverse of L and the above estimates (22),(28) and (29) we see that, for t, ε_0 sufficiently small, J_t maps B_{ε_0} to itself. Also by these estimates we get

$$\|J_t(\phi_2) - J_t(\phi_1)\|_{C^{2,\alpha}_\nu} \le (Ct + C(\varepsilon_0))\|\phi_2 - \phi_1\|_{C^{2,\alpha}_\nu} \;\; for \; all \;\; \phi_2, \phi_1 \in B_{\varepsilon_0},$$

where $C(\varepsilon_0) \to 0$ as $\varepsilon_0 \to 0$. This shows that for sufficiently small t and ε_0, $J_t : B_{\varepsilon_0} \to B_{\varepsilon_0}$ is a contraction and hence we can apply Banach's Contraction Mapping Principle to obtain a fixed point $\phi \in B_{\varepsilon_0}$. By taking $\varepsilon_0 > 0$ small enough we see that $v(x) = v_0(x) + \phi(x)$ is positive in Ω by considering (23) and satisfies (7).

5 More general equations

We now point out some more general equations that can be handled by the above method.

1. We first consider the Hénon like equation

$$\begin{cases} -\Delta u = |x|^\alpha u^p & \text{in } \Omega_t, \\ u = 0 & \text{on } \partial\Omega_t. \end{cases} \tag{30}$$

Then $v_0(r,\theta) := r^{\frac{-2-\alpha}{p-1}} w(\theta)$ is a separable positive solution on the unperturbed equation provided w is a positive classical solution of

$$-\Delta_\theta w = \hat{\lambda}_p w + w^p \quad \text{in } S, \qquad w = 0 \quad \text{on } S,$$

where

$$\hat{\lambda}_p := \left(\frac{\alpha+2}{p-1}\right)\left(\frac{2+\alpha}{p-1} - (N-2)\right).$$

Note for general S we will require that p is subcritical; hence $p < \frac{N+1}{N-3}$ and we will need $\hat{\lambda}_p < \lambda_1(S)$, where $\lambda_1(S)$ is defined as before. Note this implies various restrictions on the allowed parameters; for instance for S and p fixed we have $\hat{\lambda}_p \to \infty$ as $\alpha \to \infty$; hence for large α we have $\hat{\lambda}_p > \lambda_1(S)$. Once we find a positive classical solution w we still need to show its nondegenerate; which would then show using the same method as above that the linearized operator $L(\phi) := -\Delta\phi - \frac{pw(\theta)^{p-1}\phi}{r^2}$ is surjective and we can proceed as before.

2. We can also examine equations of the form

$$\begin{cases} -\Delta u &= u^p \pm |x|^\beta u^q \qquad \text{in } \Omega_t, \\ \quad u &= 0 \qquad\qquad\qquad \text{on } \partial\Omega_t. \end{cases} \tag{31}$$

As before we look for separable solutions on the unperturbed cone. A computation shows that for $\beta := \frac{2}{p-1}(q-1) - 2$ we have $v_0(r,\theta) = r^{\frac{-2}{p-1}} w(\theta)$ is a separable solution provided $w > 0$ a classical solution of

$$-\Delta_\theta w = \lambda_p w + w^p \pm w^q \quad \text{in } S, \qquad w = 0 \quad \text{on } \partial S,$$

where

$$\lambda_p := \frac{2}{p-1}\left(\frac{2}{p-1} - (N-2)\right).$$

To follow our approach we will need to show the linearized operator is surjective onto to Y; which follow using the same argument as before (hence we need to prove the nondegeneracy of w).

3. Consider

$$\begin{cases} -\Delta u + \frac{C(\theta)}{|x|^2} u &= u^p \qquad \text{in } \Omega_t, \\ \qquad\qquad u &= 0 \qquad \text{on } \partial\Omega_t. \end{cases} \tag{32}$$

A computation shows that $v_0(r,\theta) := r^{\frac{-2}{p-1}} w(\theta)$ is a positive singular solution of (32) on the unperturbed cone Ω provided $w > 0$ is a classical solution of

$$-\Delta_\theta w + C(\theta) w = \lambda_p w + w^p \quad \text{in } S, \qquad w = 0 \quad \text{on } \partial S.$$

As before to find positive singular solution of (32) for $t > 0$ small we will need w to be a nondegenerate solution.

6 Alternate function spaces

In this section we consider some alternate functions spaces to consider our problem in, the purpose being that this may allow one to use perturbations that are not admissible in the weighted Hölder spaces we used in the previous sections. Instead of considering perturbations of the cone we prefer here to consider lower order perturbations of the equation on the unperturbed cone Ω, where Ω is defined as in the introduction. This will be sufficient to illustrate that there are benefits for considering these new spaces. Recall in the previous sections $Y = C^{0,\alpha}_{\nu-2}$ and X was the $C^{2,\alpha}_{\nu}$ functions which were zero on $\partial\Omega$. We will now denote this second space as $C^{2,\alpha}_{\nu,0}$; any reference to spaces X and Y from here forwards refers to the spaces defined below. We now define the following new norms

$$\|f\|_Y := \sup_{s>0} s^{\alpha+1-\frac{N}{2}} \|f\|_{H^{-1}(A_s)},$$

$$\|\phi\|_{X_1} := \sup_{s>0} s^{\alpha+1-\frac{N}{2}} \|\nabla\phi\|_{L^2(A_s)}, \qquad \|\phi\|_{X_0} := \sup_{s>0} s^{\alpha-\frac{N}{2}} \|\phi\|_{L^2(A_s)}$$

which are the weighted H^{-1} and H^1_0 spaces we consider. We will also consider some weighted L^q spaces

$$\|f\|_{Y_q} := \sup_{s>0} s^{\alpha+2-\frac{N}{q}} \|f\|_{L^q(A_s)},$$

$$\|\phi\|_{X^0_q} := \sup_{s>0} s^{\alpha-\frac{N}{q}} \|\phi\|_{L^q(A_s)}, \qquad \|\phi\|_{X^1_q} := \sup_{s>0} s^{\alpha+1-\frac{N}{q}} \|\nabla\phi\|_{L^q(A_s)},$$

$$\|\phi\|_{X^2_q} := \|\phi\|_{X^1_q} + \sup_{s>0} s^{\alpha+2-\frac{N}{q}} \|D^2\phi\|_{L^q(A_s)}.$$

As before we define L via (8) where $w = w_p$ denotes a positive classical solution of (3).

We now consider two perturbations of (2) given by

$$\begin{cases} -\Delta u + V(x)u = u^p & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\backslash\{0\}, \end{cases} \tag{33}$$

and

$$\begin{cases} -\Delta u = |u|^p + f(x) & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega\backslash\{0\}. \end{cases} \tag{34}$$

Recall if $V(x) = \frac{C(\theta)}{|x|^2}$ we can handle (33) via the prior method, see Section 5. Here we would like to consider cases of $V(x)$ which our prior methods would not be able to handle. As before we look for solutions of the form $u(x) = v_0(x) + \phi(x)$ of (33); but we replace u^p with $|u|^p$ for now, and recall v_0 is the explicit separable solution of (2). We then need ϕ to satisfy

$$\begin{cases} L(\phi) = H(\phi) - V(x)\phi - V(x)v_0 & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega\backslash\{0\}, \end{cases} \tag{35}$$

where $H(\phi) := |v_0 + \phi|^p - v_0^p - pv_0^{p-1}\phi$. Using the same approach we see to find a solution of (34) we need ϕ to satisfy

$$\begin{cases} L(\phi) = H(\phi) + f(x) & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega\backslash\{0\}. \end{cases} \tag{36}$$

Before we carry on we state our main linear result regarding these new spaces.

Proposition 4. *Suppose $N \geq 3$.*

1. *Suppose $p \in (p_0, p_1)$, $\mu_k \neq 0$ for all $k \geq 1$. Then there is some $C > 0$ such that for all $f \in Y$ there is some $\phi \in X_1$ such that $L(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$. Moreover one has the estimate $\|\phi\|_{X_1} \leq C\|f\|_Y$.*

2. *Suppose $p \in (p_0, p_2)$, $1 < q < \infty$ and $\mu_k \neq 0$ for all $k \geq 1$. Then there is some C_q such that for all $f \in Y_q$ there is some $\phi \in X_q^2$ such that $L(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$. Moreover one has the estimate $\|\phi\|_{X_q^2} \leq C_q\|f\|_{Y_q}$.*

6.0.1 Equation (33)

Here we examine conditions on $V(x)$ so we can obtain a positive solution of (33). Our main interest is to weaken the spaces to allow for $V(x)$ for which the weighted Hölder spaces we used in the previous sections wouldn't work. So we want $V(x)$ to not be smooth; but we will impose smallness conditions (to see examples of $V(x)$ without smallness assumptions see Section 5). We consider $V(x) = V_\varepsilon(x) = \frac{h_\varepsilon(x)}{|x|^2}$ where $\varepsilon > 0$ is a small parameter. So we consider the nonlinear operator $J_\varepsilon(\phi) = \psi$ where

$$\left\{ \begin{array}{rcll} L(\psi) & = & H(\phi) - \frac{h_\varepsilon \phi}{|x|^2} - \frac{h_\varepsilon v_0}{|x|^2} & \text{in } \Omega, \\ \psi & = & 0 & \text{on } \partial\Omega\backslash\{0\}. \end{array} \right. \tag{37}$$

We begin with some estimates on $H(\phi)$ (see below (35) for the definition of H). For $p > 1$ there is some C (all constants just depend on p) such that for all $0 < v_0 \in \mathbb{R}$ and $\phi_i, \phi \in \mathbb{R}$ one has

$$|H(\phi)| \leq C(v_0^{p-1} + |\phi|^{p-1})|\phi|, \tag{38}$$

$$|H(\phi_2) - H(\phi_1)| \leq C\Big(v_0^{p-1} + |\phi_2|^{p-1} + |\phi_1|^{p-1}\Big)||\phi_2 - \phi_1|, \tag{39}$$

$$|H(\phi)| \leq C\Big(v_0^{p-2}\phi^2 + |\phi|^p\Big), \tag{40}$$

$$|H(\phi_2) - H(\phi_1)| \leq C\Big(v_0^{p-2}(|\phi_1| + |\phi_2|) + |\phi_1|^{p-1} + |\phi_2|^{p-1}\Big)|\phi_2 - \phi_1|. \tag{41}$$

If we restrict $1 < p \leq 2$ then we have

$$|H(\phi)| \leq C|\phi|^p \tag{42}$$

$$|H(\phi_2) - H(\phi_1)| \leq C\Big(|\phi_2|^{p-1} + |\phi_1|^{p-1}\Big)|\phi_2 - \phi_1|. \tag{43}$$

The estimates (40) and (41) are completely standard so we omit any discussion of them. For the remaining estimates see Section 7.

We now state our main theorem in this section.

Theorem 2. *Suppose S, p, N are as in the hypothesis of Theorem 1 part 1 and for some $q \in (N, \infty)$ one has*

$$\sup_{s>0} \frac{1}{s^N} \int_{A_s} |h_\varepsilon(x)|^q dx \to 0, \tag{44}$$

as $\varepsilon \searrow 0$. Then for $\varepsilon > 0$ small enough there is a positive solution of (33) where $V(x) = \frac{h_\varepsilon(x)}{|x|^2}$.

Proof. Fix p, q, S, N as in the hypothesis. Define J_ε as above. We will show that J_ε is a contraction on B_R where B_R is the closed ball centered at the origin in X_q^2. The exact approach will depend on whether $p \le 2$ or $p > 2$. We first consider the case of $p \in (p_0, 2]$. Given $\phi \in B_R$ a scaling argument and Sobolev imbedding theorems show the existence of $C > 0$ such that

$$\sup_{s<|x|<2s} |\phi| \le \frac{C\|\phi\|_{X_q^2}}{s^\alpha} \tag{45}$$

for all $s > 0$.
Into. Let $\phi \in B_R$. Using (42) we see and (45) one immediately sees that

$$\|H(\phi)\|_{Y_q} \le \sup_{s>0} \frac{s^{\alpha+2-\frac{N}{q}} C\|\phi\|_{X_q^2}^p s^{\frac{N}{q}}}{s^{\alpha p}} \le CR^p,$$

after noting the exponent on s is zero. A similar computation shows

$$\|h_\varepsilon v_0 |x|^{-2}\|_{Y_q}^q \le C \sup_{s>0} \frac{1}{s^N} \int_{A_s} |h_\varepsilon(x)|^q w(\theta)^q dx$$

which converges to zero as $\varepsilon \to 0$ by hypothesis. A similar computation and using (45) we see

$$\|h_\varepsilon \phi |x|^{-2}\|_{Y_q}^q \le CR^q \sup_{s>0} \frac{1}{s^N} \int_{A_s} |h_\varepsilon(x)|^q dx.$$

We set $T_\varepsilon := (1 + \max_S w^q) \sup_{s>0} \frac{1}{s^N} \int_{A_s} |h_\varepsilon(x)|^q dx$. Then we have $J_\varepsilon(B_R) \subset B_R$ provided

$$CR^p + CT_\varepsilon + CR^q T_\varepsilon \le R. \tag{46}$$

Contraction. Let $\phi_i \in B_R$ and then using (43) and (45) we have

$$\|H(\phi_2) - H(\phi_1)\|_{Y_q}^q \le C \sup_{s>0} s^{q(\alpha+2-\alpha p)} R^{q(p-1)} \|\phi_2 - \phi_1\|_{X_q^2}^q$$

and now note the exponent on s is zero. Now let $J_\varepsilon(\phi_i) = \psi_i$. A similar computation shows

$$\|h_\varepsilon |x|^{-2}(\phi_2 - \phi_1)\|_{Y_q} \le CT_\varepsilon^{\frac{1}{q}} \|\phi_2 - \phi_1\|_{X_q^2}.$$

Combining the above two estimates shows

$$\|\psi_2 - \psi_1\|_{X_q^2} \le \left(CR^{p-1} + CT_\varepsilon^{\frac{1}{q}} \right) \|\phi_2 - \phi_1\|_{X_q^2}$$

and hence using this we see J_ε is a contraction on B_R provided (46) holds and $CR^{p-1}+CT_\varepsilon^{\frac{1}{q}} < 1$. Note we can easily satisfy both these condition by fixing $R>0$ small and then taking $\varepsilon>0$ small. We then can apply the Banach fixed point argument to see the existence of a fixed point $\phi \in B_R$ such that $J_\varepsilon(\phi)=\phi$. By taking $R>0$ sufficiently small its clear that $u=v_0+\phi$ is nonzero and is a classical solution of (33) with $V(x)=\frac{h_\varepsilon(x)}{|x|^2}$ in the case of u^p replaced with $|u|^p$. Since $q>N$ and hence $W^{2,q} \subset C^{0,1}$ we can argue as in the case of weighted Hölder spaces to see $u>0$ and hence satisfies (33).

We now consider the case of $p>2$. In this case we use the inequality (41). First note that to estimate $\|H(\phi_2)-H(\phi_1)\|_{Y_q}$ we just need to estimate the Y_q norm of $v_0^{p-2}|\phi_i||\phi_2-\phi_1|$ for $i=1,2$, because the other terms are computed in the above case and note we did not use the restriction $1<p\le 2$ there. So we now have

$$\|v_0^{p-2}|\phi_i||\phi_2-\phi_1|\|_{Y_q}^q \le \sup_{s>0} s^{(\alpha+2)q-N} \int_{A_s} v_0^{(p-2)q}|\phi_i|^q|\phi_2-\phi_1|^q \simeq Cs^{2\alpha q-N}\int_{A_s}|\phi_i|^q|\phi_2-\phi_1|^q$$

$$\le Cs^{2\alpha q-N}\left(\sup_{s<|x|<2s}|\phi_i|\right)^q\left(\sup_{s<|x|<2s}|\phi_2-\phi_1|\right)^q s^N$$

and by (45) we have this is bounded above by

$$\le C\sup_{s>0} s^{2\alpha q}\frac{\|\phi_i\|_{X_q^2}^q}{s^{\alpha q}}\frac{\|\phi_2-\phi_1\|_{X_q^2}^q}{s^{\alpha q}} \le CR^q\|\phi_2-\phi_1\|_{X_q^2}^q.$$

Hence we proved that

$$\|v_0^{p-2}|\phi_i||\phi_2-\phi_1|\|_{Y_q} \le CR\|\phi_2-\phi_1\|_{X_q^2},$$

and hence when coupled with the other term we have

$$\|H(\phi_2)-H(\phi_1)\|_{Y_q} \le C(R+R^{p-1})\|\phi_2-\phi_1\|_{X_q^2}. \tag{47}$$

From this we see that

$$\|\psi_2-\psi_1\|_{X_q^2} \le \left(C(R+R^{p-1})+CT_\varepsilon^{\frac{1}{q}}\right)\|\phi_2-\phi_1\|_{X_q^2}$$

and hence we have J_ε a contraction on B_R (but we haven't proven into yet) provided

$$C(R+R^{p-1})+CT_\varepsilon^{\frac{1}{q}} \le \frac{3}{4}. \tag{48}$$

Now let $\phi\in B_R$ and $J_\varepsilon(\phi)=\psi$. Using the above estimate with $\phi_2=\phi$ and $\phi_1=0$ (and since $H(0)=0$) we arrive at

$$\|H(\phi)\|_{Y_q} \le C(R^2+R^p).$$

So we then have

$$\|\psi\|_{X_q^2} \leq C(R^2 + R^p) + C\||x|^{-2} h_\varepsilon \phi\|_{Y_q} + C\|x|^{-2} h_\varepsilon v_0\|_{Y_q}$$

and using the earliear estimates on the other terms we arrive at

$$\|\psi\|_{X_q^2} \leq C(R^2 + R^p) + CRT_\varepsilon^{\frac{1}{q}}$$

and hence for $J_\varepsilon(B_R) \subset B_R$ it is sufficient that

$$C(R^2 + R^p) + CRT_\varepsilon^{\frac{1}{q}} \leq R.$$

By fixing $R > 0$ sufficiently small and then taking $\varepsilon > 0$ small we see that we can satisfy this property along with (48) and hence J_ε is a contraction on B_R and now we can proceed as in the case of $p \leq 2$.

$\square$

Remark 1. *Instead of working with $q > N$ we can also work with $q = \frac{N}{2p'} \in (1, \frac{N}{2})$, $p' = \frac{p}{p-1}$, and here one needs weaker assumptions on the smallness of h_ε. The problem with this approach is one does not get for free the positivity of $u = v_0 + \phi$ (but they do get $u \neq 0$). So one still needs to prove u is positive and hence we chose to not pursue this. Also note for the problem at hand the weighted Hölder spaces we used in the beginning would not be sufficient.*

6.0.2 Equation (34)

We now state our main theorem for this section.

Theorem 3. *Suppose S, p, N are as in the hypothesis of Theorem 1 part 1 and additionally we assume $p < \frac{N+2}{N-2}$. Then for $\|f\|_Y$ small enough there exists a solution $u \in X_1$ of (34).*

Proof. Define the nonlinear mapping $J(\phi) = \psi$ via

$$\begin{cases} L(\psi) &= H(\phi) + f(x) \quad \text{in } \Omega, \\ \psi &= 0 \quad \text{on } \partial\Omega \backslash \{0\}, \end{cases} \tag{49}$$

and to obtain a solution ϕ of (36) we show J is a contraction on B_R (the closed ball of radius R centered at the origin in X_1). We fix p as above and we assume additionally that $p \leq 2$.
Into. Let $\phi \in B_R$ and let $\psi = J(\phi)$. Then we have

$$\|\psi\|_{X_1} \leq C\|f\|_Y + C\|H(\phi)\|_Y.$$

Note there is some $C > 0$ such that $\|g\|_{H^{-1}(A_s)} \leq C\|g\|_{L^{\frac{2N}{N+2}}(A_s)}$ where C independent of s. Using this we have

$$\|H(\phi)\|_{H^{-1}(A_s)} \leq C\|H(\phi)\|_{L^{\frac{2N}{N+2}}(A_s)} \leq C\|\phi\|^p_{L^{\frac{2pN}{N+2}}(A_s)}$$

and hence we have

$$\|H(\phi)\|_Y \leq C \sup_{s>0} s^{\alpha+1-\frac{N}{2}} \|\phi\|^p_{L^{\frac{2pN}{N+2}}(A_s)}.$$

For $1 < q \leq 2^*$ there is some $C > 0$ such that

$$\|\phi\|_{L^q(A_s)} \leq \frac{C\|\phi\|_{X_1}}{s^{\alpha-\frac{N}{q}}} \tag{50}$$

for all $s > 0$. From this we see that

$$\|H(\phi)\|_Y \leq C \sup_{s>0} \frac{s^{\alpha+1-\frac{N}{2}} \|\phi\|^p_{X_1}}{s^{p(\alpha-\frac{N}{q})}}$$

where we are taking $q = \frac{2pN}{N+2}$ which is in the allowable range since we have $p < \frac{N+2}{N-2}$. From the definition of q and α we see the exponent on s is zero and hence we have

$$\|H(\phi)\|_Y \leq CR^p.$$

So we see that if $\psi = J(\phi)$ that

$$\|\psi\|_{X_1} \leq C\|H(\phi)\|_Y + C\|f\|_Y \leq CR^p + C\|f\|_Y$$

and hence to have $J(B_R) \subset B_R$ we need

$$CR^p + C\|f\|_Y \leq R. \tag{51}$$

Contraction. Let $\phi_i \in B_R$ and $\psi_i = J(\phi_i)$. Then we have

$$\|\psi_2 - \psi_1\|_{X_1} \leq C\|H(\phi_2) - H(\phi_1)\|_Y$$

and we now estimate the right hand side of this using (43). So we have

$$\|H(\phi_2)-H(\phi_1)\|_{H^{-1}(A_s)} \leq C\|H(\phi_2)-H(\phi_1)\|_{L^{\frac{2N}{N+2}}(A_s)} \leq C\|(|\phi_1|^{p-1}+|\phi_2|^{p-1})|\phi_2-\phi_1|\|_{L^{\frac{2N}{N+2}}(A_s)} =: I$$

and we now examine one term of I;

$$I^{\frac{2N}{N+2}} = \int_{A_s} |\phi_1|^{\frac{(p-1)2N}{N+2}} |\phi_2 - \phi_1|^{\frac{2N}{N+2}} dx$$

and we now apply Höler's inequality on this with $\frac{2Nt}{N+2} = 2^* = \frac{2N}{N-2}$ to obtain

$$I \leq \|\phi_1\|^{p-1}_{L^{\frac{N(p-1)}{2}}(A_s)} \|\phi_2 - \phi_1\|_{L^{2^*}(A_s)}$$

and note since $p < \frac{N+2}{N-2}$ we have $\frac{N(p-1)}{2} < 2^*$. We now use (50) to see

$$I \leq \frac{CR^{p-1}\|\phi_2 - \phi_1\|_{X_1}}{s^{\alpha p-1-\frac{N}{2}}}$$

and from this we see

$$\|H(\phi_2) - H(\phi_1)\|_Y \leq \left(CR^{p-1} \sup_{s>0} s^{\alpha+2-\alpha p} \right) \|\phi_2 - \phi_1\|_{X_1}$$

and noting the exponent on s is zero, and hence we have

$$\|\psi_2 - \psi_1\|_{X_1} \leq CR^{p-1} \|\phi_2 - \phi_1\|_{X_1}.$$

Using this and (51) we see for small $R > 0$ and then assuming $\|f\|_Y$ is small we see that J is a contraction on B_R and hence we can apply Banach's fixed point theorem. The case of $p > 2$ is similar; again we omit the details.

□

6.1 The linear theory

In this section we prove Proposition 4. We begin by examining the kernel of L in these various new spaces.

Lemma 3. *(Kernel of L) Suppose $N \geq 3$.*

1. *Suppose $p \in (p_0, p_1)$, $\mu_k \neq 0$ for all $k \geq 1$ and $\phi \in X_1$ is such that $L(\phi) = 0$ in Ω with $\phi = 0$ on $\partial\Omega$. Then $\phi = 0$.*

2. *Suppose $p \in (p_0, p_2)$, $1 < q < \infty$ and $\mu_k \neq 0$ for all $k \geq 1$. Suppose $\phi \in X_q^2$ is such that $L(\phi) = 0$ in Ω with $\phi = 0$ on $\partial\Omega$. Then $\phi = 0$.*

Proof. Let $\phi \in X_1$ or X_q^2 be such that $L(\phi) = 0$ with $\phi = 0$ on $\partial\Omega$. We now write ϕ as $\phi(x) = \sum_{k=1}^{\infty} a_k(r)\psi_k(\theta)$. We now obtain bounds on each $a_k(r)$. First multiply the infinite sum representation of ϕ by $\psi_k(\theta)$ and integrate over S one obtains (after taking absolute values) that

$$|a_k(r)| \leq \int_S |\phi(r\theta)||\psi_k(\theta)| d\theta \leq T_k \int_S |\phi(r\theta)| d\theta$$

where T_k is a constant depending on k, and then we can apply Jensen's inequality to see that

$$|a_k(r)|^q \leq \tilde{T}_k \int_S |\phi(r\theta)|^q d\theta$$

and hence we have

$$\int_s^{2s} r^{N-1} |a_k(r)|^q dr \leq T_k \int_{A_s} |\phi(x)|^q dx \leq \frac{T_k C}{s^{q\alpha - N}} \tag{52}$$

for all $s > 0$; here $q > 1$. Note this estimate gives

$$\int_s^{2s} \frac{1}{r} |a_k(r) r^\alpha|^q dr \leq T_k. \tag{53}$$

From this we see

$$\inf_{s<r<2s} |r^{\alpha} a_k(r)|^q \le \tilde{T}_k \tag{54}$$

for all $s > 0$.

We now obtain the form of $a_k(r)$. Note that $a_k(r)$ satisfies the following

$$0 = \sum_{k=1}^{\infty} \left(-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + (\nu(N-2+\nu) + \mu_k) \frac{a_k(r)}{r^2} \right) \psi_k(\theta), \tag{55}$$

and hence we must have

$$-a_k''(r) - \frac{(N-1)a_k'(r)}{r} + (\nu(N-2+\nu) + \mu_k) \frac{a_k(r)}{r^2} = 0 \quad 0 < r < \infty, \tag{56}$$

which are ode's of Euler type. Define

$$\gamma_k := -(\mu_k + \nu(N-2+\nu)),$$

and hence we can re-write the above ode's as

$$a_k''(r) + \frac{(N-1)}{r} a_k(r) + \gamma_k \frac{a_k(r)}{r^2} = 0 \quad 0 < r < \infty.$$

Looking for solutions of the form $a(r) = r^{\beta}$ we see β needs to satisfy

$$\beta^2 + (N-2)\beta + \gamma_k = 0,$$

and from this we need to consider three cases:
Case I: $(N-2)^2 - 4\gamma_k > 0$
Case II: $(N-2)^2 - 4\gamma_k = 0$
Case III: $(N-2)^2 - 4\gamma_k < 0$.

Case I. In this case we have

$$\beta_k^+ := \frac{-(N-2)}{2} + \frac{\sqrt{(N-2+2\nu)^2 + 4\mu_k}}{2}$$

$$\beta_k^- := \frac{-(N-2)}{2} - \frac{\sqrt{(N-2+2\nu)^2 + 4\mu_k}}{2},$$

and hence $a_k(r) = C_k r^{\beta_k^+} + D_k r^{\beta_k^-}$ for some $C_k, D_k \in \mathbb{R}$ and note $\beta_k^- < \beta_k^+$. To show $C_k = D_k = 0$ we claim it will be sufficient to show that both of β_k^+, β_k^- differ from ν. In this case we have

$$a_k(r) r^{\alpha} = C_k r^{\beta_k^+ + \alpha} + D_k r^{\beta_k^- + \alpha}$$

and note both exponents are nonzero by hypothesis. If either of C_k or D_k is nonzero we can obtain a contradiction by considering (54) and sending $s \to 0$ or $s \to \infty$.

Case II. In this case we have

$$|r^\alpha a_k(r)| = r^{\alpha - \frac{N-2}{2}} \Big| C_k + D_k \ln r \Big|$$

And since $\alpha \neq \frac{N-2}{2}$ (because $p \neq \frac{N+2}{N-2}$) we see that if $C_k \neq 0$ or $D_k \neq 0$ then $\lim_{r\to 0} |r^\alpha a_k(r)| = \infty$ or $\lim_{r\to\infty} |r^\alpha a_k(r)| = \infty$, a contradiction, hence $C_k = D_k = 0$.

Case III. $(N-2)^2 - 4\gamma_k < 0$. In this case define

$$\omega_k := \frac{\sqrt{4\gamma_k - (N-2)^2}}{2}$$

and then the general solution given by

$$a_k(r) = C_k r^{\frac{2-N}{2}} \sin(\omega_k \ln(r)) + D_k r^{\frac{2-N}{2}} \cos(\omega_k \ln(r)),$$

for constants C_k, D_k. Now note we have

$$\frac{\tilde{T}_k}{s^{q\alpha - N}} \geq \int_s^{2s} r^{N-1} |a_k(r)|^q dr = \int_s^{2s} r^{N-1+q\frac{2-N}{2}} \Big| C_k \sin(\omega_k \ln(r)) + D_k \cos(\omega_k \ln(r)) \Big|^q dr.$$

Changing the variable $\tau = \frac{r}{s}$ in the last integral we get

$$\begin{aligned} \frac{\tilde{T}_k}{s^{q(\alpha - \frac{N-2}{2})}} &\geq \int_1^2 \tau^{N-1+q\frac{2-N}{2}} \Big| C_k \sin(\omega_k \ln(s\tau)) + D_k \cos(\omega_k \ln(s\tau)) \Big|^q d\tau \\ &= (C_k^2 + D_k^2)^{\frac{q}{2}} \int_1^2 \tau^{N-1+q\frac{2-N}{2}} \Big| \sin(\omega_k \ln(s\tau) + \theta_k \Big|^q d\tau, \end{aligned}$$

where $\theta_k := \arcsin \frac{D_k}{\sqrt{C_k^2 + D_k^2}}$. Now taking $s = s_m = e^{\pm \frac{2m\pi}{\omega_k}}$, $m = 1, 2, ...$ we get

$$\frac{\tilde{T}_k}{s_m^{q(\alpha - \frac{N-2}{2})}} \geq (C_k^2 + D_k^2)^{\frac{q}{2}} \int_1^2 \tau^{N-1+q\frac{2-N}{2}} \Big| \sin(\omega_k \ln(\tau) + \theta_k \Big|^q d\tau = A_k > 0, \ m = 1, 2, ...$$

which is impossible by letting $m \to \infty$ noting that $\alpha - \frac{N-2}{2} \neq 0$, unless we have $C_k = D_k = 0$. $\square$

Proof of Proposition 4. The idea of the proof in 1 and 2 will be to prove the result for $f \in C^{0,\alpha}_{\nu-2}$ (and hence $\phi \in C^{2,\alpha}_{\nu,0}$), and then perform a blow up argument. This will give us the desired result, but for this reduced class of f. We then will extend to the full space of f. We proof part 2 first since we don't have to deal with H^{-1} in this case.

2. So we fix $1 < q < \infty$ and we claim there is some C_q such that for all $f \in C^{0,\alpha}_{\nu-2}$ there is some $\phi \in C^{2,\alpha}_{\nu,0}$ with $L(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$ and we have $\|\phi\|_{X^0_q} \leq C_q \|f\|_{Y_q}$. Of course the existence is not an issue, only the estimate might be false. So if we assume

the result is false then there is $f_m \in C^{0,\alpha}_{\nu-2}$ and $\phi_m \in C^{2,\alpha}_{\nu,0}$ such that $L(\phi_m) = f_m$ in Ω with $\phi_m = 0$ on $\partial\Omega$ and $\|\phi_m\|_{X^0_q} = 1$ and $\|f_m\|_{Y_q} \to 0$. Note after rescaling and using boundary elliptic regularity theory we see that ϕ_m is bounded in X^2_q (to see a similar calculation see below). Also note there is some $s_m > 0$ such that $s_m^{q\alpha-N}\|\phi_m\|^q_{L^q(A_{s_m})} \geq \frac{3}{4}$. We now define $\zeta_m(x) := s_m^\alpha \phi_m(s_m x)$ and note $\int_{1<|x|<2} |\zeta_m(x)|^q dx \geq \frac{3}{4}$. Also note a computation shows that for any i we have

$$\int_{2^i<|x|<2^{i+1}} |\zeta_m(x)|^q dx \leq 2^{i(N-q\alpha)}. \tag{57}$$

Also note we have

$$-\Delta\zeta_m(x) - \frac{pw(\theta)^{p-1}\zeta_m(x)}{|x|^2} = s_m^{2+\alpha} f_m(s_m x) =: g_m(x) \qquad \text{in } \Omega \tag{58}$$

with $\zeta_m = 0$ on $\partial\Omega$. For large integers k we set $E_k := \{x \in \Omega : \frac{1}{k} < |x| < k\}$ and $\tilde{E}_k := \{x \in \Omega : \frac{1}{2k} < |x| < 2k\}$. A computation shows that $g_m \to 0$ in $L^q(\tilde{E}_k)$. Also note for each large k there is some C_k such that we have

$$\|\zeta_m\|_{W^{2,q}(E_k)} \leq C_k \|\Delta\zeta_m\|_{L^q(\tilde{E}_k)} + C_k \|\zeta_m\|_{L^q(\tilde{E}_k)} \tag{59}$$

which shows there is some $\tilde{C}_k$ such that $\|\zeta_m\|_{W^{2,q}(E_k)} \leq C$ for all m. By passing to a subsequence and using a diagonal argument we can assume $\zeta_m \rightharpoonup \zeta$ in $W^{2,q}(E_k)$ for any k large and ζ satisfies $L(\zeta) = 0$ in Ω with $\zeta = 0$ on $\partial\Omega$. Now also note we can pass to the limit in (57) and also in the inequality above (57). From this we see that $\zeta \in X^0_q$ is nonzero. As before a scaling argument shows that $\zeta \in X^2_q$ and hence we have a contradiction to the kernel of L being trivial; see Lemma 3. We have now proved the initial claim; we need to extend this to the full set of f.
Fix $f \in Y_q$ and for large m we set $f_m(x) = f(x)$ for $x \in E_m$ with $f_m = 0$ otherwise. Then we have $\|f^m\|_{Y_q} \leq \|f\|_{Y_q}$. By density there is some $f_m \in C^{0,\alpha}_{\nu-2}$ (in fact zero near the vertex of the cone and ∞) such that $\|f_m - f^m\|_{Y_q} \leq m^{-1}\|f^m\|_{Y_q}$ and hence we have

$$\|f_m\|_{Y_q} \leq \|f_m - f^m\|_{Y_q} + \|f^m\|_{Y_q} \leq (1+m^{-1})\|f\|_{Y_q} \leq 2\|f\|_{Y_q}$$

and for any large integer k there is some C_k such that

$$\|f_m - f^m\|_{L^q(\tilde{E}_k)} \leq C_k \|f_m - f^m\|_{Y_q} \leq \frac{C_k}{m}\|f^m\|_{Y_q} \leq \frac{C_k}{m}\|f\|_{Y_q}.$$

Let $L(\phi_m) = f_m$ in Ω with $\phi_m = 0$ on $\partial\Omega$. Then by the above estimates we have $\|\phi_m\|_{X^2_q} \leq C\|f_m\|_{Y_q} \leq 2C\|f\|_{Y_q}$ and hence we can pass to a subsequence and find some ϕ such that $\phi_m \rightharpoonup \phi \in W^{2,q}(E_k)$ for all k and then note we have $L(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$. We now fix $s > 0$ and note we can pass to the limit to see

$$s^{q\alpha-N} \int_{A_s} |\phi(x)|^q dx \leq 2^q C^q \|f\|^q_{Y_q}$$

and now we can sup over s to see that we have the desired X_q^0 estimate on ϕ. We can now use scaling to get the full X_q^2 bound on ϕ.

1. We claim there is some $C > 0$ such that for all $f \in C^{0,\alpha}_{\nu-2}$ there is some $\phi \in C^{2,\alpha}_{\nu,0}$ with $L(\phi) = f$ in Ω with $\phi = 0$ on $\partial\Omega$ and we have $\|\phi\|_{X_0} \le C\|f\|_Y$ (note that for any ϕ as above we must have $\phi \in H^1(\tilde{E}_k)$ for all k). As before the only issue is the estimate and so if we assume its false there is some ϕ_m and f_m which satisfy the equation and $\|\phi_m\|_{X_0} = 1$, $\|f_m\|_Y \to 0$. Then there is some $s_m > 0$ such that

$$s_m^{2\alpha-N} \int_{A_{s_m}} |\phi_m(x)|^2 dx \ge \frac{3}{4}.$$

Set $\zeta_m(x) := s_m^\alpha \phi_m(s_m x)$ and note for all i we have

$$\int_{2^i<|x|<2^{i+1}} |\zeta_m(x)|^2 dx \le 2^{i(N-2\alpha)} \qquad \text{and} \qquad \int_{1<|x|<2} |\zeta_m(x)|^2 dx \ge \frac{3}{4}. \tag{60}$$

As before we have ζ_m satisfies (58). We now make a few claims that we will prove later;
Claim 1. For each large k, g_m (as defined in (58)) satisfies $\|g_m\|_{H^{-1}(\tilde{E}_k)} \to 0$ as $m \to \infty$.
Claim 2. There C_k such that for all $\psi \in H^1_{0,loc}(\Omega)$ (by this we mean $\psi \in H^1(E_k)$ for all k and $\psi = 0$, in the sense of trace, on the lateral boundary of E_k for each k) we have

$$\|\nabla\psi\|_{L^2(E_k)} \le C_k \|\Delta\psi\|_{H^{-1}(\tilde{E}_k)} + C_k \|\psi\|_{L^2(\tilde{E}_k)}.$$

Using Claim 1 and Claim 2 we see there is some C_k such that $\|\zeta_m\|_{H^1(E_k)} \le C_k$ for all m and using a diagonal argument there is some $\zeta \in H^1_{0,loc}(\Omega)$ such that $\zeta_m \rightharpoonup \zeta$ in $H^1(E_k)$ for all k and ζ satisfies $L(\zeta) = 0$ in Ω with $\zeta = 0$ on $\partial\Omega$. Fix $s > 0$ and we can pass to the limit in (60) to see ζ is nonzero and

$$\int_{2^i<|x|<2^{i+1}} |\zeta(x)|^2 dx \le 2^{i(N-2\alpha)}.$$

Using this bound we see $\zeta \in X_0$. Fix $s > 0$ and set $\zeta_s(x) = \zeta(sx)$ and $-\Delta\zeta_s(x) = |x|^{-2} p w(\theta)^{p-1} \zeta_s(x)$ in Ω with $\zeta_s = 0$ on $\partial\Omega$ and so we can use Claim 2 with $k = 2$ to get

$$s^{2\alpha+2-N} \int_{s<|y|<2s} |\nabla\zeta(y)|^2 dy \le C s^{2\alpha-N} \int_{4^{-1}s<|y|<4s} |\zeta(y)|^2 dy \le C\|\zeta\|^2_{X_0}$$

and we can now take the supremum over s; this gives us $\zeta \in X_1$. This gives us the desired contradiction after recalling the kernel of L is trivial.

As before we now extend the estimate to the full space Y. For m large let γ_m be a piecewise linear, Lipschitz cut off with $\gamma_m = 1$ for $m^{-1} < |x| < m$ with $\gamma_m = 0$ for $|x| < (2m)^{-1}$ or $|x| > 2m$; so there is some C such that $|\nabla\gamma_m| \le Cm^{-1}$ in $m < |x| < 2m$ and $|\nabla\gamma_m| \le Cm$ in $2^{-1}m^{-1} < |x| < m^{-1}$.

We now define f^m by $f_m = \gamma_m f$ which is well defined (recall f may not be a function). We now show that there is some C (independent of f and $m \geq 2$)) such that $\|f^m\|_Y \leq C\|f\|_Y$. Fix $s > 0$ and note

$$|\langle f^m, \phi\rangle| \leq \|f\|_{H^{-1}(A_s)}\|\nabla(\gamma_m\phi)\|_{L^2(A_s)}$$

for all $\phi \in H_0^1(A_s)$ where $\langle\cdot,\cdot\rangle$ denote the $H^{-1}(A_s)$, $H_0^1(A_s)$ duality pairing. Using the gradient bounds on γ_m and noting the first eigenvalue of $-\Delta$ on $H_0^1(A_s)$ we see there is some $C > 0$ (independent of s, m and f) such that $\|\nabla(\gamma_m\phi)\|_{L^2(A_s)} \leq C\|\nabla\phi\|_{L^2(A_s)}$ and from this we see $\|f^m\|_{H^{-1}(A_s)} \leq C\|f\|_{H^{-1}(A_s)}$ and hence we have $\|f^m\|_Y \leq C\|f\|_Y$. For each m as above there is some $f_m \in C^{0,\alpha}_{\nu-2}$ (in fact its zero near the vertex of the cone and near ∞) such that $\|f_m - f^m\|_Y \leq m^{-1}\|f^m\|_Y \leq m^{-1}C\|f\|_Y$ after considering the above estimate. Then we get

$$\|f_m\|_Y \leq \|f_m - f^m\|_Y + \|f^m\|_Y \leq (1+m^{-1})C\|f\|_Y \leq 2C\|f\|_Y$$

for all large m. Note if we write the earlier inequality to see that for all $s > 0$ that

$$s^{\alpha+1-\frac{N}{2}}\|f_m - f^m\|_{H^{-1}(A_s)} \leq Cm^{-1}\|f\|_Y.$$

We will show for each k there is some C_k (depending only on k) such that

$$\|f_m - f^m\|_{H^{-1}(\tilde{E}_k)} \leq C_k \sup_{\frac{k}{2}\leq s\leq k}\|f_m - f^m\|_{H^{-1}(A_s)} \leq \tilde{C}_k\|f_m - f^m\|_Y \leq \frac{\tilde{C}_k}{m}\|f\|_Y.$$

The only inequality in the above that we need to justify is $\|f_m - f^m\|_{H^{-1}(\tilde{E}_k)} \leq C_k \sup_{\frac{k}{2}\leq s\leq k}\|f_m - f^m\|_{H^{-1}(A_s)}$. To prove this result it will be sufficient to prove the following: there is some C (independent of f) such that for any $f \in Y$ we have

$$\|f\|_{H^{-1}(A_{1,3})} \leq C\|f\|_{H^{-1}(A_{1,2})} + C\|f\|_{H^{-1}(A_{\frac{3}{2},3})}$$

where $A_{a,b} := \{x \in \Omega : a < |x| < b\}$. Fix some smooth $0 \leq \gamma$ on $\mathbb{R}^N$ with $\gamma = 1$ in $B_{\frac{3}{2}}$ with $\gamma \in C_c^\infty(B_2)$. Then note for $\phi \in H_0^1(A_{1,3})$ we have $\phi = \gamma\phi + (1-\gamma)\phi$. For $0 < a < b$ we write $\langle\cdot,\cdot\rangle_{A_{a,b}}$ to be the $H^{-1}(A_{a,b})$, $H_0^1(A_{a,b})$ duality pairing. Then we have

$$\left|\langle f, \phi\rangle_{A_{1,3}}\right| \leq \|f\|_{H^{-1}(A_{1,2})}\|\nabla(\gamma\phi)\|_{L^2(A_{1,2})} + \|f\|_{H^{-1}(A_{\frac{3}{2},3})}\|\nabla((1-\gamma)\phi)\|_{L^2(A_{\frac{3}{2},3})}.$$

The L^2 norms on the right hand side of the above inequality are all bounded above by $C\|\nabla\phi\|_{L^2(A_{1,3})}$ where C depends on the cut off γ. Now taking a supremum over ϕ gives the desired result.

Proof of Claim 1. We first show that $\|g_m\|_{H^{-1}(A_k)} \to 0$ for any $k > 0$. Fix $k > 0$ and let $-\Delta v_m = g_m$ in A_k with $v_m \in H_0^1(A_k)$ and so $\|g_m\|_{H^{-1}(A_k)} = \|\nabla v_m\|_{L^2(A_k)}$.

Set $\hat{v}_m(x) := s_m^{-\alpha}v_m(s_m^{-1}x)$ for $ks_m < |x| < 2ks_m$ and hence $-\Delta\hat{v}_m(x) = f_m(x)$ in A_{ks_m} with $\hat{v}_m = 0$ on ∂A_{ks_m}. Hence we have $\|f_m\|_{H^{-1}(A_{ks_m})} = \|\nabla\hat{v}_m\|_{L^2(A_{ks_m})}$. From this we see

$$\begin{aligned}
\int_{k<|y|<2k}|\nabla v_m(y)|^2dy &= s_m^{2\alpha+2-N}\int_{ks_m<|x|<2ks_m}|\nabla\hat{v}_m(x)|^2dx \\
&= k^{N-2\alpha-2}(ks_m)^{2\alpha+2-N}\|f_m\|_{H^{-1}(A_{ks_m})} \\
&\leq k^{N-2\alpha-2}\|f_m\|_Y \to 0,
\end{aligned}$$

as $m \to \infty$ and hence $\|g_m\|_{H^1(A_k)} \to 0$ as $m \to \infty$. To see we can extend this to $\tilde{E}_k$ we take $k = 1$ and $k = \frac{3}{2}$ and use the above argument to see that $\|g_m\|_{H^{-1}(A_{1,3})} \to 0$. We can keep doing this process of joining intersecting annular regions of the form A_s and obtaining H^{-1} estimates on the union. Hence we see for any fixed k we have $\|g_m\|_{H^{-1}(\tilde{E}_k)} \to 0$ as $m \to \infty$. This completes the proof of Claim 1.

Proof of Claim 2. The proof follows the exact proof one would use for the L^p version of this estimate; which gives $W^{2,p}(\tilde{E}_k)$ bounds.

□

7 Appendix

Here we collect some of the proofs for the estimates (38)-(43).

Lemma 4. *If $1 < p \leq 2$, then we have*

$$0 \leq H(\phi) \leq c_p|\phi|^p \tag{61}$$

and

$$|H(\phi_2) - H(\phi_1)| \leq C_p\Big(|\phi_2|^{p-1} + |\phi_1|^{p-1}\Big)|\phi_2 - \phi_1|. \tag{62}$$

Proof. First note we have the known inequalities

$$\Big(|b|^{p-2}b - |a|^{p-2}a\Big)(b-a) > 0, \quad a \neq b \quad for\ all\ p > 1 \tag{63}$$

$$0 \leq \Big(|b|^{p-2}b - |a|^{p-2}a\Big)(b-a) \leq c_p|b-a|^p, \quad 1 < p \leq 2, \tag{64}$$

see, for example, [31] for the above computations. Now we write by the mean value theorem (note $|t|^p$ is differentiable when $p > 1$)

$$H(\phi) = H(\phi) - H(0) = \phi H'(z),$$

for some $z = t\phi$, $t \in [0,1]$, and then

$$H(\phi) = \phi H'(z) = \frac{H'(z)z}{t} = \frac{p}{t}\Big(|v_0 + z|^{p-2}(v_0+z) - v_0^{p-2}v_0\Big)(v_0 + z - v_0)$$

Hence using (63) and (64) we get $H(\phi) \geq 0$ for all $p > 1$ and also

$$H(\phi) \leq c_p t^{p-1}|\phi|^p \leq c_p|\phi|^p \quad 1 < p \leq 2,$$

that proves (61). To see (62) we have

$$H(\phi_2) - H(\phi_1) = (\phi_2 - \phi_1)H'(z),$$

where $z = \theta\phi_1 + (1-\theta)\phi_2$ for some $\theta \in [0,1]$, hence

$$H(\phi_2) - H(\phi_1) = (\phi_2 - \phi_1)\frac{H'(z)z}{z},$$

and then using (64) again, we obtain

$$|H(\phi_2) - H(\phi_1)| = \frac{|\phi_2 - \phi_1|}{|z|}H'(z)z \leq C_p|\phi_2 - \phi_1||z|^{p-1} \leq C(|\phi_2|^{p-1} + |\phi_1|^{p-1})|\phi_2 - \phi_1|,$$

that proves (62). □

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